Appendix

Appendix A

– Appendix A1

The coefficients in equations (10-13):

$$A_{11} = \frac{E_1}{1-\nu^2} + \mu_1 \frac{b_x}{d_x} E_1^{sx} + 2\mu_2 \frac{b_l}{d_l} E_1^{sl} \cos^4\theta,\ A_{12} = \frac{\nu E_1}{1-\nu^2} + 2\mu_2 \frac{b_l}{d_l} E_1^{sl} \sin^2\theta \cos^2\theta,$$

$$A_{22} = \frac{E_1}{1-\nu^2} + \mu_1 \frac{b_y}{d_y} E_1^{sy} + 2\mu_2 \frac{b_l}{d_l} E_1^{sl} \sin^4\theta,$$

$$A_{66} = \frac{E_1}{2(1+\nu)} + 2\mu_2 \frac{b_l}{d_l} E_1^{sl} \sin^2\theta \cos^2\theta,$$

$$B_{11} = \frac{E_2}{1-\nu^2} + \mu_1 \frac{b_x}{d_x} E_2^{sx} + 2\mu_2 \frac{b_l}{d_l} E_2^{sl} \cos^4\theta,$$

$$B_{12} = \frac{\nu E_2}{1-\nu^2} + 2\mu_2 \frac{b_l}{d_l} E_2^{sl} \sin^2\theta \cos^2\theta,$$

$$B_{22} = \frac{E_2}{1-\nu^2} + \mu_1 \frac{b_y}{d_y} E_2^{sy} + 2\mu_2 \frac{b_l}{d_l} E_2^{sl} \sin^4\theta,$$

$$B_{66} = \frac{E_2}{2(1+\nu)} + 2\mu_2 \frac{b_l}{d_l} E_2^{sl} \sin^2\theta \cos^2\theta,$$

$$C_{11} = \frac{E_4}{1-\nu^2} + \mu_1 \frac{b_x}{d_x} E_4^{sx} + 2\mu_2 \frac{b_l}{d_l} E_4^{sl} \cos^4\theta,$$

$$C_{12} = \frac{\nu E_4}{1-\nu^2} + 2\mu_2 \frac{b_l}{d_l} E_4^{sl} \sin^2\theta \cos^2\theta,$$

$$C_{22} = \frac{E_4}{1-\nu^2} + \mu_1 \frac{b_y}{d_y} E_4^{sy} + 2\mu_2 \frac{b_l}{d_l} E_4^{sl} \sin^4\theta,$$

$$C_{66} = \frac{E_4}{2(1+\nu)} + 2\mu_2 \frac{b_l}{d_l} E_4^{sl} \sin^2\theta \cos^2\theta,$$

$$D_{11} = \frac{E_3}{1-\nu^2} + \mu_1 \frac{b_x}{d_x} E_3^{sx} + 2\mu_2 \frac{b_l}{d_l} E_3^{sl} \cos^4\theta,$$

$$D_{12} = \frac{\nu E_3}{1-\nu^2} + 2\mu_2 \frac{b_l}{d_l} E_3^{sl} \sin^2\theta \cos^2\theta,$$

$$D_{22}=\frac{E_3}{1-\nu^2}+\mu_1\frac{b_y}{d_y}E_3^{sy}+2\mu_2\frac{b_l}{d_l}E_3^{sl}\sin^4\theta,$$

$$D_{66}=\frac{E_3}{2(1+\nu)}+2\mu_2\frac{b_l}{d_l}E_3^{sl}\sin^2\theta\cos^2\theta,$$

$$E_{11}=\frac{E_5}{1-\nu^2}+\mu_1\frac{b_x}{d_x}E_5^{sx}+2\mu_2\frac{b_l}{d_l}E_5^{sl}\cos^4\theta,$$

$$E_{12}=\frac{\nu E_5}{1-\nu^2}+2\mu_2\frac{b_l}{d_l}E_5^{sl}\sin^2\theta\cos^2\theta,$$

$$E_{22}=\frac{E_5}{1-\nu^2}+\mu_1\frac{b_y}{d_y}E_5^{sy}+2\mu_2\frac{b_l}{d_l}E_5^{sl}\sin^4\theta,$$

$$E_{66}=\frac{E_5}{2(1+\nu)}+2\mu_2\frac{b_l}{d_l}E_5^{sl}\sin^2\theta\cos^2\theta,$$

$$L_{11}=\frac{E_7}{1-\nu^2}+\mu_1\frac{b_x}{d_x}E_7^{sx}+2\mu_2\frac{b_l}{d_l}E_7^{sl}\cos^4\theta,$$

$$L_{12}=\frac{\nu E_7}{1-\nu^2}+2\mu_2\frac{b_l}{d_l}E_7^{sl}\sin^2\theta\cos^2\theta,$$

$$L_{22}=\frac{E_7}{1-\nu^2}+\mu_1\frac{b_y}{d_y}E_7^{sy}+2\mu_2\frac{b_l}{d_l}E_7^{sl}\sin^4\theta,$$

$$L_{66}=\frac{E_7}{2(1+\nu)}+2\mu_2\frac{b_l}{d_l}E_7^{sl}\sin^2\theta\cos^2\theta,$$

$$\overline{H}_{44}=\frac{E_1}{2(1+\nu)}+\mu_1\frac{b_x}{d_x}\frac{E_1^{sx}}{2(1+\nu)}+\mu_2 2\sin^2\theta\frac{1}{2(1+\nu)}\frac{b_l}{d_l}E_1^{sl}-$$
$$-3\lambda\left[\frac{E_3}{2(1+\nu)}+\mu_1\frac{b_x}{d_x}\frac{E_3^{sx}}{2(1+\nu)}+\mu_2 2\sin^2\theta\frac{1}{2(1+\nu)}\frac{b_l}{d_l}E_3^{sl}\right],$$

$$\overline{H}_{55}=\frac{E_1}{2(1+\nu)}+\mu_1\frac{b_y}{d_y}\frac{E_1^{sy}}{2(1+\nu)}+\mu_2 2\cos^2\theta\frac{1}{2(1+\nu)}\frac{b_l}{d_l}E_1^{sl}-$$
$$-3\lambda\left[\frac{E_3}{2(1+\nu)}+\mu_1\frac{b_y}{d_y}\frac{E_3^{sy}}{2(1+\nu)}+\mu_2 2\cos^2\theta\frac{1}{2(1+\nu)}\frac{b_l}{d_l}E_3^{sl}\right],$$

$$H_{66}=\frac{E_3}{2(1+\nu)}+\mu_1\frac{b_x}{d_x}\frac{E_3^{sx}}{2(1+\nu)}+\mu_2 2\sin^2\theta\frac{1}{2(1+\nu)}\frac{b_l}{d_l}E_3^{sl}-$$
$$-3\lambda\left[\frac{E_5}{2(1+\nu)}+\mu_1\frac{b_x}{d_x}\frac{E_5^{sx}}{2(1+\nu)}+\mu_2 2\sin^2\theta\frac{1}{2(1+\nu)}\frac{b_l}{d_l}E_5^{sl}\right],$$

$$H_{77}=\frac{E_3}{2(1+\nu)}+\mu_1\frac{b_y}{d_y}\frac{E_3^{sy}}{2(1+\nu)}+\mu_2 2\cos^2\theta\frac{1}{2(1+\nu)}\frac{b_l}{d_l}E_3^{sl}-$$
$$-3\lambda\left[\frac{E_5}{2(1+\nu)}+\mu_1\frac{b_y}{d_y}\frac{E_5^{sy}}{2(1+\nu)}+\mu_2 2\cos^2\theta\frac{1}{2(1+\nu)}\frac{b_l}{d_l}E_5^{sl}\right].$$

$(E_1,E_2,E_3,E_4,E_5,E_7)=\int_{-h/2}^{h/2}\left(1,z,z^2,z^3,z^4,z^6\right)E_p(z)dz$ defined as:

+ For the first case:

$$E_1=E_c h+E_{mc}\left(h_c+\frac{h_t}{k_t+1}+\frac{h_b}{k_b+1}\right),$$

$$E_2=E_{mc}\left[\frac{h_c(h_t-h_b)}{2}+\frac{h_t^2}{k_t+2}-\frac{hh_t}{2(k_t+1)}-\frac{h_b^2}{k_b+2}+\frac{hh_b}{2(k_b+1)}\right],$$

$$E_3=E_c\frac{h^3}{12}+E_{mc}\left[\frac{(h-2h_b)^3+(h-2h_t)^3}{24}+\frac{h_t^3}{k_t+3}-\frac{hh_t^2}{k_t+2}+\frac{h^2h_t}{4(k_t+1)}+\right.$$
$$\left.+\frac{h_b^3}{k_b+3}-\frac{hh_b^2}{k_b+2}+\frac{h^2h_b}{4(k_b+1)}\right],$$

$$E_4=E_{mc}\left[\frac{(h-2h_b)^4-(h-2h_t)^4}{64}+\frac{h_t^4}{k_t+4}-\frac{3hh_t^3}{2(k_t+3)}+\frac{3h^2h_t^2}{4(k_t+2)}-\right.$$
$$\left.-\frac{h^3h_t}{8(k_t+1)}-\frac{h_b^4}{k_b+4}+\frac{3hh_b^3}{2(k_b+3)}-\frac{3h^2h_b^2}{4(k_b+2)}+\frac{h^3h_b}{8(k_b+1)}\right],$$

$$E_5=E_c\frac{h^5}{80}+E_{mc}\left[\frac{(h-2h_b)^5+(h-2h_t)^5}{24}+\frac{h_t^5}{k_t+5}-\frac{2hh_t^4}{k_t+4}+\frac{3h^2h_t^3}{2(k_t+3)}-\right.$$
$$\left.-\frac{h^3h_t^2}{2(k_t+2)}-\frac{h^4h_t}{16(k_t+1)}+\frac{h_b^5}{k_b+5}-\frac{2hh_b^4}{k_b+4}+\frac{3h^2h_b^3}{2(k_b+3)}-\frac{h^3h_b^2}{2(k_b+2)}-\frac{h^4h_b}{16(k_b+1)}\right],$$

$$E_7=E_c\frac{h^7}{448}+E_{mc}\left[\frac{(h-2h_b)^7+(h-2h_t)^7}{896}+\frac{h_t^7}{k_t+7}-\frac{3hh_t^6}{k_t+6}+\frac{15h^2h_t^5}{4(k_t+5)}-\right.$$
$$-\frac{5h^3h_t^4}{2(k_t+4)}+\frac{15h^4h_t^3}{16(k_t+3)}-\frac{3h^5h_t^2}{16(k_t+2)}+\frac{h^6h_t}{64(k_t+1)}+\frac{h_b^7}{k_b+7}-\frac{3hh_b^6}{k_b+6}+$$
$$\left.+\frac{15h^2h_b^5}{4(k_b+5)}-\frac{5h^3h_b^4}{2(k_b+4)}+\frac{15h^4h_b^3}{16(k_b+3)}-\frac{3h^5h_b^2}{16(k_b+2)}+\frac{h^6h_b}{64(k_b+1)}\right].$$

$$\left(E_1^{si}, E_2^{si}, E_3^{si}, E_4^{si}, E_5^{si}, E_7^{si}\right) = \int_{h/2}^{h/2+h_i} \left(1, z, z^2, z^3, z^4, z^6\right) E_{si}(z)dz; i = x, y, l \text{ defined as:}$$

$$E_1^{si} = E_c h_i + E_{mc} h_i \frac{1}{k_i + 1},$$

$$E_2^{si} = \frac{E_c}{2} h_i (h + h_i) + E_{mc} h_i^2 \left(\frac{1}{k_i + 2} + \frac{h}{2h_i} \frac{1}{k_i + 1} \right),$$

$$E_3^{si} = \frac{E_c}{3} \left[\left(\frac{h}{2} + h_i \right)^3 + \frac{h^3}{8} \right] + E_{mc} h_i^3 \left(\frac{1}{k_i + 3} + \frac{h}{h_i} \frac{1}{k_i + 2} + \frac{h^2}{4h_i^2} \frac{1}{k_i + 1} \right),$$

$$E_4^{si} = \frac{E_c}{4} \left[\left(\frac{h}{2} + h_i \right)^4 - \frac{h^4}{16} \right] + E_{mc} h_i^4 \left(\frac{1}{k_i + 4} + \frac{3h}{2h_i} \frac{1}{k_i + 3} + \frac{3h^2}{4h_i^2} \frac{1}{k_i + 2} + \frac{h^3}{8h_i^3} \frac{1}{k_i + 1} \right),$$

$$E_5^{si} = \frac{E_c}{5} \left[\left(\frac{h}{2} + h_i \right)^5 - \frac{h^5}{32} \right] + E_{mc} h_i^5 \left(\frac{1}{k_i + 5} + \frac{2h}{h_i} \frac{1}{k_i + 4} + \frac{3h^2}{2h_i^2} \frac{1}{k_i + 3} + \frac{h^3}{2h_i^3} \frac{1}{k_i + 2} + \frac{h^4}{16h_i^4} \frac{1}{k_i + 1} \right),$$

$$E_7^{si} = \frac{E_c}{7} \left[\left(\frac{h}{2} + h_i \right)^7 - \frac{h^7}{128} \right] + E_{mc} h_i^7 \left(\frac{1}{k_i + 7} + \frac{3h}{h_i} \frac{1}{k_i + 6} + \frac{15h^2}{4h_i^2} \frac{1}{k_i + 5} + \frac{5h^3}{2h_i^3} \frac{1}{k_i + 4} + \frac{15h^4}{16h_i^4} \frac{1}{k_i + 3} + \frac{3h^5}{16h_i^5} \frac{1}{k_i + 2} + \frac{h^6}{64h_i^6} \frac{1}{k_i + 1} \right).$$

+ For the second case:

The expressions of E_i are similar to the first case by replacing E_c by E_m and E_{mc} by E_{cm}.

– Appendix A2

The coefficients in equations (14-16):

$$X_{11} = \frac{A_{22}}{A_{11}A_{22} - A_{12}^2},$$

$$X_{12} = \frac{A_{12}}{A_{11}A_{22} - A_{12}^2},$$

$$X_{13} = B_{11}X_{11} - B_{12}X_{12},$$

$$X_{14} = B_{12}X_{11} - B_{22}X_{12},$$

$$X_{15} = C_{11}X_{11} - C_{12}X_{12},$$

$$X_{16} = C_{12}X_{11} - C_{22}X_{12},$$

$$X_{17} = X_{11} - X_{12},$$

$$X_{18} = X_{11},$$

$$X_{19} = -X_{12},$$

$$X_{21} = \frac{A_{11}}{A_{11}A_{22} - A_{12}^2},$$

$$X_{22} = B_{12}X_{21} - B_{11}X_{12},$$

$$X_{23} = B_{22}X_{21} - B_{12}X_{12}$$

$$X_{24} = C_{12}X_{21} - C_{11}X_{12},$$

$$X_{25} = C_{22}X_{21} - C_{12}X_{12},$$

$$X_{26} = X_{21} - X_{12},$$

$$X_{27} = -X_{12},$$

$$X_{28} = X_{21},$$

$$X_{31} = \frac{1}{A_{66}},$$

$$X_{32} = \frac{B_{66}}{A_{66}},$$

$$X_{33} = \frac{C_{66}}{A_{66}},$$

$$Y_{11} = D_{11} - B_{11}X_{13} - B_{12}X_{22},$$

$$Y_{12} = D_{12} - B_{11}X_{14} - B_{12}X_{23},$$

$$Y_{13} = E_{11} - B_{11}X_{15} - B_{12}X_{24},$$

$$Y_{14} = E_{12} - B_{11}X_{16} - B_{12}X_{25},$$

$$Y_{15} = B_{11}X_{17} + B_{12}X_{26},$$

$$Y_{16} = B_{11}X_{18} + B_{12}X_{27},$$

$$Y_{17} = B_{11}X_{19} + B_{12}X_{28},$$

$$Y_{21} = D_{12} - B_{12}X_{13} - B_{22}X_{22},$$

$$Y_{22} = D_{22} - B_{22}X_{23} - B_{12}X_{14},$$

$$Y_{23} = E_{12} - B_{12}X_{15} - B_{22}X_{24},$$

$$Y_{24} = E_{22} - B_{22}X_{25} - B_{12}X_{16},$$

$$Y_{25} = B_{12}X_{17} + B_{22}X_{26},$$

$$Y_{26} = B_{12}X_{18} + B_{22}X_{27},$$

$$Y_{27} = B_{12}X_{19} + B_{22}X_{28},$$

$$Y_{31} = D_{66} - B_{66}X_{32},$$

$$Y_{32} = E_{66} - B_{66}X_{33},$$

$$Z_{11} = L_{11} - C_{11}X_{15} - C_{12}X_{24},$$

$$Z_{12} = L_{12} - C_{11}X_{16} - C_{12}X_{25},$$

$$Z_{13} = C_{11}X_{17} + C_{12}X_{26},$$

$$Z_{14} = C_{11}X_{18} + C_{12}X_{27},$$

$$Z_{15} = C_{11}X_{19} + C_{12}X_{28},$$

$$Z_{21} = L_{12} - C_{12}X_{15} - C_{22}X_{24},$$

$$Z_{22} = L_{22} - C_{22}X_{25} - C_{12}X_{16},$$

$$Z_{23} = C_{12}X_{17} + C_{22}X_{26},$$

$$Z_{24} = C_{12}X_{18} + C_{22}X_{27},$$

$$Z_{25} = C_{12}X_{19} + C_{22}X_{28},$$

$$Z_{31} = L_{66} - C_{66}X_{33}.$$

Appendix B

The coefficients in system of equation (26)

$$\kappa_{11} = -\lambda^2\left[Z_{11}\alpha^4 + \left(Z_{12} + Z_{21} + 4Z_{31}\right)\alpha^2\beta^2 + Z_{22}\beta^4\right] - K_1 - \\ - K_2\left(\alpha^2 + \beta^2\right) - \left[\left(\overline{H}_{44} - 3\lambda H_{66}\right)\alpha^2 + \left(\overline{H}_{55} - 3\lambda H_{77}\right)\beta^2\right] - DM_3^2,$$

$$\kappa_{12} = \lambda\left\{\left[\left(Y_{23} + 2Y_{32}\right) - \lambda\left(Z_{21} + 2Z_{31}\right)\right]\alpha\beta^2 + \left(Y_{13} - \lambda Z_{11}\right)\alpha^3\right\} - \\ - DM_1M_3 - \left(\overline{H}_{44} - 3\lambda H_{66}\right)\alpha,$$

$$\kappa_{13} = \lambda\left\{\left[\left(Y_{14} + 2Y_{32}\right) - \lambda\left(Z_{12} + 2Z_{31}\right)\right]\alpha^2\beta + \left(Y_{24} - 2Z_{22}\right)\beta^3\right\} - \\ - DM_2M_3 - \left(\overline{H}_{55} - 3\lambda H_{77}\right)\beta,$$

$$\kappa_{14} = M_1 \frac{8\alpha^2\beta^2}{3mn\pi^2}\delta_m\delta_n, \quad \kappa_{15} = M_2 \frac{8\alpha^2\beta^2}{3mn\pi^2}\delta_m\delta_n,$$

$$\kappa_{21} = \lambda\left\{\left[\left(Y_{14} + 2Y_{32}\right) - \lambda\left(Z_{12} + 2Z_{31}\right)\right]\alpha\beta^2 + \left(Y_{13} - \lambda Z_{11}\right)\alpha^3\right\} -$$
$$- DM_1M_3 - \left(\overline{H}_{44} - 3\lambda H_{66}\right)\alpha,$$

$$\kappa_{22} = -\left[Y_{11} - 2\lambda Y_{13} + \lambda^2 Z_{11}\right]\alpha^2 - \left[Y_{31} - 2\lambda Y_{32} + \lambda^2 Z_{31}\right]\beta^2 -$$
$$-\left(\overline{H}_{44} - 3\lambda H_{66}\right) - DM_1^2,$$

$$\kappa_{23} = -\left[Y_{12} + Y_{31} - 2\lambda\left(Y_{14} + Y_{32}\right) + \lambda^2\left(Z_{12} + Z_{31}\right)\right]\alpha\beta - DM_1M_2,$$

$$\kappa_{31} = \lambda\left\{\left[\left(Y_{23} + 2Y_{32}\right) - \lambda\left(Z_{21} + 2Z_{31}\right)\right]\alpha^2\beta + \left(Y_{24} - \lambda Z_{22}\right)\beta^3\right\} -$$
$$- DM_2M_3 - \left(\overline{H}_{55} - 3\lambda H_{77}\right)\beta,$$

$$\kappa_{32} = -\left[Y_{21} + Y_{31} - 2\lambda\left(Y_{23} + Y_{32}\right) + \lambda^2\left(Z_{21} + Z_{31}\right)\right]\alpha\beta - DM_1M_2,$$

$$\kappa_{33} = -\left[Y_{31} - 2\lambda Y_{32} + \lambda^2 Z_{31}\right]\alpha^2 - \left[Y_{22} - 2\lambda Y_{24} + \lambda^2 Z_{22}\right]\beta^2 -$$
$$-\left(\overline{H}_{55} - 3\lambda H_{77}\right) - DM_2^2,$$

$$\varsigma_1 = M_3 \frac{8\alpha^2\beta^2}{3mn\pi^2}\delta_m\delta_n, \quad \varsigma_2 = -\left(\frac{\lambda X_{24}}{X_{21}} + \frac{\lambda X_{16}}{X_{11}}\right)\frac{2\alpha^2\beta^2}{3mn\pi^2}\delta_m\delta_n,$$

$$\varsigma_3 = -\frac{1}{16}\left(\frac{\alpha^4}{X_{11}} + \frac{\beta^4}{X_{21}}\right), \quad \varsigma_4 = \frac{4}{mn\pi^2}\delta_m\delta_n,$$

$$\varsigma_5 = -\frac{\left(X_{22} - \lambda X_{24}\right)}{X_{21}}\frac{2\alpha\beta^2}{3mn\pi^2}\delta_m\delta_n,$$

$$\varsigma_6 = -\frac{\left(X_{14} - \lambda X_{16}\right)}{X_{11}}\frac{2\alpha^2\beta}{3mn\pi^2}\delta_m\delta_n, \quad \varsigma_7 = -\frac{9K_3}{16}$$

in which

$$\delta_m = \cos(m\pi) - 1, \delta_n = \cos(n\pi) - 1.$$

Appendix C

The coefficients in equation (33)

$$\tau_1 = \frac{1}{X_{21}}\left[\tau_{11} + \frac{\left(\kappa_{23}\kappa_{31} - \kappa_{33}\kappa_{21}\right)\tau_{12} + \left(\kappa_{32}\kappa_{21} - \kappa_{22}\kappa_{31}\right)\tau_{13}}{\kappa_{22}\kappa_{33} - \kappa_{23}\kappa_{32}}\right],$$

$$\tau_2 = \frac{1}{X_{21}}\left[\tau_{14} + \frac{\left(\kappa_{23}\varsigma_6 - \kappa_{33}\varsigma_5\right)\tau_{12} + \left(\kappa_{32}\varsigma_5 - \kappa_{22}\varsigma_6\right)\tau_{13}}{\kappa_{22}\kappa_{33} - \kappa_{23}\kappa_{32}}\right],$$

in which

$$\tau_{11} = \frac{\delta_m\delta_n}{mn\pi^2}\left\{M_3\left(X_{21}\alpha^2 - X_{12}\beta^2\right) + \lambda X_{24}\alpha^2 + \lambda X_{25}\beta^2\right\},$$
$$\tau_{12} = \frac{\delta_m\delta_n}{mn\pi^2}\left\{M_1\left(X_{21}\alpha^2 - X_{12}\beta^2\right) - \left(X_{22} - \lambda X_{16}\right)\alpha\right\},$$
$$\tau_{13} = \frac{\delta_m\delta_n}{mn\pi^2}\left\{M_2\left(X_{21}\alpha^2 - X_{12}\beta^2\right) - \left(X_{23} - \lambda X_{25}\right)\beta\right\},\ \tau_{14} = \frac{1}{8}\beta^2.$$