Appendix B

Smearing estimator for various transformations

The smearing estimator takes the following forms for the different transformation functions:
a) log

It can be easily shown that for the case of logarithmic transformation the smearing estimator becomes

$\hat{E}\left(Y \mid X\right)=\exp\left(\hat{\alpha}+\hat{\beta}_1 \cdot X_1+\ldots+\hat{\beta}_k \cdot X_k\right)\sum\exp\left(\varepsilon_i\right)$, where $\sum\exp\left(\varepsilon_i\right)$ is called the *smearing factor*[1].

b) logit

The smearing estimator becomes

$$\hat{E}\left(Y \mid X\right)=\frac{1}{n}\sum_{i=1}^{n}\operatorname{expit}\left(\hat{\alpha}+\hat{\beta}_1 \cdot X_1+\ldots+\hat{\beta}_k \cdot X_k+\varepsilon_i\right)$$

$$=\frac{1}{n}\sum_{i=1}^{n}\frac{\exp\left(\hat{\alpha}+\hat{\beta}_1 \cdot X_1+\ldots+\hat{\beta}_k \cdot X_k\right)\cdot\exp\left(\varepsilon_i\right)}{\exp\left(\hat{\alpha}+\hat{\beta}_1 \cdot X_1+\ldots+\hat{\beta}_k \cdot X_k\right)\cdot\exp\left(\varepsilon_i\right)+1}.$$

c) For the arcsin transformation, we need to consider the fact that $\cos 2\theta=1-2\sin^2\theta$, as well as the fact that $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$. We then have

$$\hat{E}\left(Y \mid X\right)=\frac{1}{n}\sum_{i=1}^{n}\sin^2\left(\frac{\hat{\alpha}+\hat{\beta}_1 \cdot X_1+\ldots+\hat{\beta}_k \cdot X_k+\varepsilon_i}{2}\right)$$

$$=\frac{1}{n}\sum_{i=1}^{n}\frac{1}{2}\left(1-\sin\left(\hat{\alpha}+\hat{\beta}_1 \cdot X_1+\ldots+\hat{\beta}_k \cdot X_k+\varepsilon_i\right)\right),$$

,which finally gives

$$\hat{E}\left(Y \mid X\right)=\frac{1}{2}\left[1-\cos\left(\hat{\alpha}+\hat{\beta}_1 \cdot X_1+\ldots+\hat{\beta}_k \cdot X_k\right)\frac{\sum_{i=1}^{n}\cos(\varepsilon_i)}{n}+\sin\left(\hat{\alpha}+\hat{\beta}_1 \cdot X_1+\ldots+\hat{\beta}_k \cdot X_k\right)\frac{\sum_{i=1}^{n}\sin(\varepsilon_i)}{n}\right].$$

d) For the Box-Cox transformation, the smearing estimator is

$$\hat{E}(Y \mid X) = \frac{1}{n} \sum_{i=1}^{n} \left[\lambda \cdot (\hat{\alpha} + \hat{\beta}_1 \cdot X_1 + \ldots + \hat{\beta}_k \cdot X_k + \varepsilon_i) + 1 \right]^{1/\lambda}$$

It is apparent that for logit and Box-Cox transformations, the smearing estimator cannot be computed as simply as in the log case, while for the arcsin transformation the complexity is somewhat intermediate.

References

1. Duan N. Smearing estimate: a nonparametric retransformation method. *Journal of the American Statistical Association*. 1983: 78(383):605-610.