APPENDIX-A

Proof of Theorem 2.1. Taking expected value of the observed response Z_i with p.m.f in (2.1), we have

$$E(Z_i) = \pi E(X) + (1-\pi)E(Y)$$

$$= \pi\left(\frac{kp_1}{1-q_1^k}\right) + (1-\pi)\left(\frac{kp_2}{1-q_2^k}\right) \tag{A.1}$$

Taking expected value on both sides of (2.9) and using (A.1), we have

$$E(\hat{\pi}_{TB}) = E\left[\frac{(1-q_1^k)(1-q_2^k)\frac{1}{n}\sum_{i=1}^{n} Z_i - k\,p_2(1-q_1^k)}{k\{p_1(1-q_2^k) - p_2(1-q_1^k)\}}\right]$$

$$= \frac{\pi k p_1(1-q_2^k) + kp_2(1-\pi)(1-q_1^k) - k\,p_2(1-q_1^k)}{k\{p_1(1-q_2^k) - p_2(1-q_1^k)\}} = \pi$$

which proves the theorem.

Proof of Theorem 2.2. Since the expected responses of Z_i are i.i.d, then by the definition of variance, we note that $V(\hat{\pi}_{TB})$ is given by

$$V(\hat{\pi}_{TB}) = V\left[\frac{[1-q_1^k][1-q_2^k]\frac{1}{n}\sum_{i=1}^{n} Z_i - k\,p_2[1-q_1^k]}{k[p_1(1-q_2^k) - p_2(1-q_1^k)]}\right]$$

$$= \frac{[1-q_1^k]^2[1-q_2^k]^2\,\frac{1}{n^2}\sum_{i=1}^{n} V(Zi)}{[k\{p_1(1-q_2^k) - p_2(1-q_1^k)\}]^2} \tag{A.2}$$

Now we have

$$V(Z_i) = E(Z_i^2) - (E(Z_i))^2$$

$$= [\pi E(X^2) + (1-\pi)E(Y^2)] - [\pi E(X) + (1-\pi)E(Y)]^2 \tag{A.3}$$

Note that an equivalent expression for $E(X^2)$ and $E(Y^2)$ are respectively, given by

$$E(X^2) = V(X) + (E(X))^2, \tag{A.4}$$

and

$$E(Y^2) = V(Y) + (E(Y))^2 \tag{A.5}$$

Substituting (A.4) and (A.5) into (A.3), for both $E(X^2)$ and $E(Y^2)$, gives us the following:

$$V(Z_i) = \pi[V(X) + (E(X))^2] + (1-\pi)[V(Y) + (E(Y))^2]$$

$$-[\pi^2(E(X))^2 + (1-\pi)^2(E(Y))^2 - 2\pi(1-\pi)E(X)E(Y)] \tag{A.6}$$

Upon substituting (A.6) into (A.2), we get the following:

$$V(\hat{\pi}_{TB}) = \frac{\frac{(1-q_1^k)^2(1-q_2^k)^2}{n^2}\sum_{i=1}^{n}\left[\pi(1-\pi)\{E(X)-E(Y)\}^2 + \pi\{E(X^2)-(E(X))^2\} + (1-\pi)\{E(Y^2)-(E(Y))^2\}\right]}{[k\{p_1(1-q_2^k) - p_2(1-q_1^k)\}]^2}$$

which on simplification proves the theorem.

ONLINE PUBLISHABLE MATERIAL

APPENDIX – B

```
#R Code used in the simulation study.
p_kuk <- .7
t_kuk <- .4
k <- 3
cat("pi","p","t","REsg", "REkuk", "RPkuk", "\n")
for(pi in seq(.05,.5,.05)){
  for(p in seq(.6,.9,.1)){
    for(t in seq(.1,.6,.1))
    {
      ex1 <- k * p / (1-(1-p)^k)
      ex2 <- k * t / (1-(1-t)^k)
      ezig <- (pi/p) + (1-pi)/t
      ezig2 <- (pi*(2-p))/(p^2) + ((1-pi)*(2-t))/(t^2)
      denog <- (1/p - 1/t)
      varg <- (ezig2 - ezig^2)/(denog^2)
      ezin <- pi*(k*p/(1-(1-p)^k)) + (1-pi)*(k*t/(1-(1-t)^k))
      ezin2 <- pi*(k*p*(1-p) + k^2 * p^2)/(1-(1-p)^k) + (1-pi)*(k*t*(1-t) + k^2 * t^2)/(1-(1-t)^k)
      denon <- (k*p/(1-(1-p)^k)) - (k*t/(1-(1-t)^k))
      varn <- (ezin2 - ezin^2)/(denon^2)

      REsg <- varg*100 / varn
```

```
      thkuk <- pi*p_kuk + (1-pi)*t_kuk
      varkuk <- thkuk*(1-thkuk) / (k*(p_kuk-t_kuk)^2) + pi*(1-pi)*(1-1/k)
       REkuk <- varkuk*100 / varn
      PAkuk = pi*k*p_kuk / (pi*k*p_kuk + (1-pi)*k*t_kuk)
      PACkuk <- (1-pi)*k*t_kuk / (pi*k*p_kuk + (1-pi)*k*t_kuk)
      PROTkuk <- max(PAkuk, PACkuk)

      PAtb <- pi*ex1 / (pi*ex1 + (1-pi)*ex2)
      PACtb <- (1-pi)*ex2 / (pi*ex1 + (1-pi)*ex2)
      PROTtb <- max(PAtb,PACtb)
      RPkuk <- PROTkuk*100 / PROTtb
      PAsg <- (pi/p) / (pi/p + (1-pi)/t)
      PACsg <- ((1-pi)/t) / (pi/p + (1-pi)/t)
      PROTsg <- max(PAsg,PACsg)
      RPsg <- PROTsg*100 / PROTtb
      if(REsg >= 100 && REkuk >= 100 && RPkuk >= 100 && RPsg >= 100 && p != t){
        cat(pi,p,t,REsg,REkuk,RPkuk, RPsg, "\n")
      }
    }
  }
}
```

Table 3.1. Percent REs and RPs for different choices of P and T with a given choice of and $\theta_2 = 0.4$.

π	P	T	RE(sg)	RE(kuk)	RP(kuk)	RP(sg)
0.05	0.70	0.10	759.23	583.26	100.96	109.44
0.05	0.70	0.20	498.87	301.63	100.03	107.62
0.05	0.80	0.10	995.50	795.79	102.10	110.77
0.05	0.80	0.20	687.50	457.19	101.05	108.92
0.05	0.80	0.30	516.87	274.38	100.08	107.18
0.05	0.90	0.10	1240.45	1022.10	$\theta_1 = 0.7$ 103.33	112.19
0.05	0.90	0.20	897.52	640.56	102.16	110.28
0.05	0.90	0.30	691.30	416.50	101.07	108.48
0.05	0.90	0.40	554.50	272.01	100.08	106.80
0.10	0.70	0.10	539.88	437.83	101.86	119.76
0.10	0.70	0.20	407.92	262.14	100.05	115.83
0.10	0.80	0.10	678.82	572.07	104.05	122.58
0.10	0.80	0.20	539.97	380.82	102.03	118.57
0.10	0.80	0.30	441.54	249.89	100.15	114.84
0.10	0.90	0.10	815.81	707.89	106.43	125.58

0.10	0.90	0.20	677.63	511.93	104.17	121.43
0.10	0.90	0.30	570.85	365.86	102.07	117.57
0.10	0.90	0.40	486.73	255.17	100.15	113.99
0.15	0.70	0.10	421.65	360.28	102.69	131.10
0.15	0.70	0.20	345.47	235.35	100.07	124.69
0.15	0.80	0.10	521.00	462.04	105.87	135.58
0.15	0.80	0.20	447.08	333.62	102.94	129.03
0.15	0.80	0.30	386.12	232.33	100.22	123.03
0.15	0.90	0.10	617.93	563.68	109.32	140.33
0.15	0.90	0.20	550.02	438.91	106.04	133.55
0.15	0.90	0.30	489.11	332.63	103.01	127.33
0.15	0.90	0.40	435.05	243.05	100.21	121.62
0.20	0.70	0.10	347.08	312.12	103.47	143.61
0.20	0.70	0.20	299.56	215.97	100.09	134.29
0.20	0.80	0.10	425.89	397.06	107.57	149.95
0.20	0.80	0.20	382.79	301.79	103.79	140.42
0.20	0.80	0.30	343.34	219.21	100.28	131.80
0.20	0.90	0.10	503.09	482.01	112.03	156.68
0.20	0.90	0.20	466.38	392.60	107.79	146.80
0.20	0.90	0.30	429.77	309.63	103.88	137.84
0.20	0.90	0.40	394.17	234.19	100.27	129.73
0.25	0.70	0.10	295.26	279.35	104.20	157.49
0.25	0.70	0.20	264.03	201.28	100.11	144.73
0.25	0.80	0.10	361.82	354.54	109.17	165.94
0.25	0.80	0.20	335.28	279.09	104.58	152.85
0.25	0.80	0.30	309.06	209.15	100.34	141.22
0.25	0.90	0.10	427.76	430.39	114.56	174.90
0.25	0.90	0.20	407.05	361.30	109.44	161.32
0.25	0.90	0.30	384.50	293.23	104.69	149.19
0.25	0.90	0.40	360.84	227.73	100.33	138.36
0.30	0.70	0.10	256.70	255.64	104.89	172.97
0.30	0.70	0.20	235.42	189.75	100.13	156.12
0.30	0.80	0.10	315.30	324.90	110.66	183.82
0.30	0.80	0.20	298.39	262.29	105.33	166.49
0.30	0.80	0.30	280.70	201.28	100.40	151.36
0.30	0.90	0.10	374.26	395.70	111.76	186.68
0.30	0.90	0.20	362.52	339.39	110.98	177.32
0.30	0.90	0.30	348.62	281.42	105.46	161.49

0.30	0.90	0.40	332.97	223.14	100.39	147.57
0.35	0.70	0.10	226.50	237.72	100.53	181.31
0.35	0.70	0.20	211.59	180.44	100.15	168.59
0.35	0.80	0.20	268.59	249.54	100.08	171.32
0.35	0.80	0.30	256.60	195.05	100.45	162.32
0.35	0.90	0.40	309.14	220.08	100.44	157.41
0.40	0.60	0.10	156.91	166.17	100.34	167.71
0.40	0.60	0.20	144.02	114.84	105.49	160.28
0.40	0.70	0.30	181.10	123.23	105.11	151.83
0.40	0.80	0.40	227.23	140.01	104.95	146.17
0.40	0.90	0.50	280.85	167.23	105.08	142.40
0.40	0.90	0.60	272.70	115.52	104.30	134.10
0.45	0.60	0.10	139.74	156.91	100.30	149.91
0.45	0.60	0.20	130.16	110.25	104.89	139.97
0.45	0.70	0.30	166.18	120.16	104.55	131.47
0.45	0.80	0.40	211.52	138.64	104.41	125.84
0.45	0.90	0.50	265.38	168.60	104.52	122.05
0.45	0.90	0.60	260.53	117.56	110.08	120.98
0.50	0.60	0.10	124.60	148.99	100.27	135.06
0.50	0.60	0.20	117.48	106.06	104.32	122.95
0.50	0.70	0.30	152.32	117.34	104.03	114.43
0.50	0.80	0.40	196.78	137.54	103.90	108.84
0.50	0.90	0.50	250.88	170.54	104.00	105.06
0.50	0.90	0.60	248.78	119.78	108.92	102.69