Web-based Supplementary Materials for “Statistical Tests for latent class in censored data due to detection limit” by

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September 9, 2019

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Web Appendix: Derivative of (3.15)

Suppose $y_i \mid \mathbf{x}_i \sim$ i.d. mTobit $(\omega, \mu_i, \sigma^2, L)$ with $\mu_i = \mathbf{x}_i^T \beta$, so the likelihood function for the i^{th} subject can be written as

$$L_i = \left[\omega + (1-\omega)\Phi\left(\frac{L-\mu_i}{\sigma}\right)\right]^{r_i}\left[(1-\omega)\frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(y_{i-}\mu_i)^2}{2\sigma^2}\right)\right]^{(1-r_i)}. \tag{S.1}$$

Then, the log-likelihood function for the whole sample is expressed by

$$l(\omega,\beta,\sigma) = \sum\nolimits_{i=1}^{n}\left\{r_i \log\left[\omega + (1-\omega)\Phi\left(\frac{L-\mu_i}{\sigma}\right)\right] + (1-r_i)\left[\log(1-\omega) - \log\left(\sqrt{2\pi}\sigma\right) - \frac{(y_{i-}\mu_i)^2}{2\sigma^2}\right]\right\}. \tag{S.2}$$

Differentiating the log-likelihood in (S.2) with respect to β, σ and ω gives

$$\frac{dl(\cdot)}{d\beta_r} = \sum_{i=1}^{n}\left\{-\frac{r_i}{\varphi}(1-\omega)A\frac{x_{ir}}{\sigma} + (1-r_i)\frac{y_{i-}\mu_i}{\sigma^2}x_{ir}\right\}, r = 1, 2, \ldots, p, \tag{S.3}$$

$$\frac{dl(\cdot)}{d\sigma} = \sum_{i=1}^{n}\left\{-\frac{r_i}{\varphi}A(1-\omega)\frac{L-\mu_i}{\sigma^2} + (1-r_i)\left[-\frac{1}{\sigma} + \frac{(y_{i-}\mu_i)^2}{\sigma^3}\right]\right\}, \tag{S.4}$$

$$\frac{dl(\cdot)}{d\omega} = \sum_{i=1}^{n}\left\{\frac{r_i(1-\varphi) - (1-r_i)\varphi}{(1-\omega)\varphi}\right\} = \sum_{i=1}^{n}\left\{\frac{r_i - \varphi}{(1-\omega)\varphi}\right\}, \tag{S.5}$$

where $A_i = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(L-\mu_i)^2}{2\sigma^2}\right)$, and $\varphi_i = \omega + (1-\omega)\Phi\left(\frac{L-\mu_i}{\sigma}\right)$.

Under $H_0 : \omega = 0$, with $\widehat{\beta}$ and $\widehat{\sigma}$ be the maximum likelihood estimates of β and α, the score function (S.3) becomes

$$\sum_{i=1}^{n}\left\{(1-r_i)\frac{y_{i-}\widehat{\mu}_i}{\widehat{\sigma}} - r_i\frac{\widehat{A}_i}{\widehat{B}_i}\right\}\frac{x_{ir}}{\widehat{\sigma}} = 0, r = 1, 2, \ldots, p, \tag{S.6}$$

(S.4) becomes

$$\sum_{i=1}^{n}\left\{(1-r_i)\left[-1 + \frac{(y_{i-}\widehat{\mu}_i)^2}{\widehat{\sigma}^2}\right] - r_i\frac{\widehat{A}_i}{\widehat{B}_i}\frac{L-\widehat{\mu}_i}{\widehat{\sigma}}\right\}\frac{1}{\widehat{\sigma}} = 0, \tag{S.7}$$

and (S.5) becomes $\sum_{i=1}^{n}\left\{\frac{r_i-\widehat{B}_i}{\widehat{B}_i}\right\}$, where $\widehat{A}_i = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(L-\widehat{\mu}_i)^2}{2\widehat{\sigma}^2}\right)$ and $\widehat{B}_i = \Phi\left(\frac{L-\widehat{\mu}_i}{\widehat{\sigma}}\right)$. Therefore, under H_0, $\widehat{\beta}$ and $\widehat{\sigma}$, the score function becomes

$$U^T(\widehat{\beta},\widehat{\sigma},0) = \left(0,0,...,0,0,\sum_{i=1}^{n}\left\{\frac{r_i-\widehat{B}_i}{\widehat{B}_i}\right\}\right). \tag{S.8}$$

$U^T(\widehat{\beta},\widehat{\sigma},0)$ is a vector with $(p+2)$ elements, the first p zeros are for the score values of $\widehat{\beta}_1,\widehat{\beta}_2,...,\widehat{\beta}_p$, and the $(p+1)^{th}$ zero is for the score of $\widehat{\sigma}$.

To simplify the notation, for all the A_i, B_i and φ_i, the subscript i was omitted in the following expressions.

The second derivatives of S.3, S.4 and S.5 with respect to β,σ and ω are given by

$$\begin{aligned}
\frac{d^2l(\cdot)}{d\beta_r d\beta_s} &= \sum_{i=1}^{n}\left[-\frac{r_i}{\varphi^2}(1-\omega)^2A^2 - \frac{r_i}{\varphi}(1-\omega)A\frac{(L-\mu_i)}{\sigma} - (1-r_i)\right]\frac{x_{ir}x_{is}}{\sigma^2}, r=1,2,\ldots,p, \quad s=1,2,\ldots,p,\\
\frac{d^2l(\cdot)}{d\beta_r d\sigma} &= \sum_{i=1}^{n}-\frac{r_i x_{ir}}{\sigma^2}\left[\frac{(1-\omega)^2A^2}{\varphi^2}\frac{L-\mu_i}{\sigma} + \frac{(1-\omega)A}{\varphi}\left(\frac{(L-\mu_i)^2}{\sigma^2}-1\right)\right] - 2(1-r_i)\frac{y_i-\mu_i}{\sigma^3}x_{ir}, r=1,2,\ldots,p\\
\frac{d^2l(\cdot)}{d\beta_r d\omega} &= \sum_{i=1}^{n}\frac{r_i A}{\varphi^2}\frac{x_{ir}}{\sigma}, r=1,2,\ldots,p,
\end{aligned}$$

and

$$\begin{aligned}
\frac{d^2l(\cdot)}{d\sigma d\sigma} &= \sum_{i=1}^{n}-r_i\frac{A(1-\omega)(L-\mu_i)}{\varphi\sigma^3}\left[\frac{A(1-\omega)}{\varphi}\frac{L-\mu_i}{\sigma} + \frac{(L-\mu_i)^2}{\sigma^2} - 2\right] + (1-r_i)\left[\frac{1}{\sigma^2} - 3\frac{(y_i-\mu_i)^2}{\sigma^4}\right],\\
\frac{d^2l(\cdot)}{d\sigma d\omega} &= \sum_{i=1}^{n}r_i\frac{1}{\varphi^2}A\frac{L-\mu_i}{\sigma^2},\\
\frac{d^2l}{d\omega d\omega} &= \sum_{i=1}^{n}-\frac{r_i\left(1-2\varphi\right)+\varphi^2}{\left(1-\omega\right)^2\varphi^2}.
\end{aligned}$$

Based on

$$E[r_i] = \Pr(r_i=1) = \Pr(y_i \le L) = \omega + (1-\omega)\Phi\left(\frac{L-\mu_i}{\sigma}\right) = \varphi, \tag{S.9}$$

and

$$E[1-r_i] = 1 - \Pr(r_i = 1) = \Pr(y_i > L) = 1 - \omega - (1-\omega)\Phi\left(\frac{L-\mu_i}{\sigma}\right) = 1 - \varphi, \quad \text{(S.10)}$$

the expected information matrix, $\mathbf{J}(\beta, \sigma, \omega)$ has the following entries:

$$\begin{aligned}
\mathbf{J}_{r,s} &= -\mathbf{E}\left(\frac{d^2 l(\cdot)}{d\beta_r d\beta_s}\right), \\
&= \sum_{i=1}^{n} \left[\frac{(1-\omega)^2 A^2}{\varphi} + (1-\omega)A\frac{(L-\mu_i)}{\sigma} + (1-\varphi)\right] \frac{x_{ir}x_{is}}{\sigma^2}, r = 1, 2, \ldots, p, \quad s = 1, 2, \ldots, p \\
\mathbf{J}_{r,p+1} &= -\mathbf{E}\left(\frac{d^2 l(\cdot)}{d\beta_r d\sigma}\right), \\
&= \sum_{i=1}^{n} \frac{x_{ir}}{\sigma^2}\left[\frac{(1-\omega)^2 A^2}{\varphi}\frac{L-\mu_i}{\sigma} + (1-\omega)A\left(\frac{(L-\mu_i)^2}{\sigma^2} - 1\right)\right] + 2(1-\varphi)\frac{y_{i-}\mu_i}{\sigma^3}x_{ir}, r = 1, 2, \ldots, p \\
\mathbf{J}_{r,p+2} &= -\mathbf{E}\left(\frac{d^2 l(\cdot)}{d\beta_r d\sigma}\right) = -\sum_{i=1}^{n} \frac{A}{\varphi}\frac{x_{ir}}{\sigma}, r = 1, 2, \ldots, p,
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{J}_{p+1,p+1} &= -\mathbf{E}\left(\frac{d^2 l(\cdot)}{d\sigma d\sigma}\right), \\
&= \sum_{i=1}^{n} \frac{A(1-\omega)(L-\mu_i)}{\sigma^3}\left[\frac{A(1-\omega)}{\varphi}\frac{L-\mu_i}{\sigma} + \frac{(L-\mu_i)^2}{\sigma^2} - 2\right] - (1-\varphi)\left[\frac{1}{\sigma^2} - 3\frac{(y_{i-}\mu_i)^2}{\sigma^4}\right], \\
\mathbf{J}_{p+1,p+2} &= -\mathbf{E}\left(\frac{d^2 l(\cdot)}{d\sigma d\omega}\right) = -\sum_{i=1}^{n} \frac{A}{\varphi}\frac{L-\mu_i}{\sigma^2}, \\
\mathbf{J}_{p+2,p+2} &= -\mathbf{E}\left(\frac{d^2 l}{d\omega d\omega}\right) = \sum_{i=1}^{n} \frac{(1-2\varphi)+\varphi}{(1-\omega)^2\varphi}.
\end{aligned}$$

So the $\mathbf{J}(\widehat{\beta}, \widehat{\sigma}, 0)$ has entries

$$
\begin{aligned}
\widehat{\mathbf{J}}_{r,s} &= \sum_{i=1}^{n} \left[\frac{\widehat{A}^2}{\widehat{B}} + \widehat{A} \frac{(L - \widehat{\mu}_i)}{\widehat{\sigma}} + 1 - B \right] \frac{x_{ir} x_{is}}{\widehat{\sigma}^2}, r = 1, 2, \ldots, p, \quad s = 1, 2, \ldots, p, \\
\widehat{\mathbf{J}}_{r,p+1} &= \sum_{i=1}^{n} \frac{x_{ir}}{\widehat{\sigma}^2} \left[\frac{\widehat{A}^2}{\widehat{B}} \frac{L - \widehat{\mu}_i}{\widehat{\sigma}} + \widehat{A} \frac{(L - \widehat{\mu}_i)^2}{\widehat{\sigma}^2} - \widehat{A} + 2 (1 - B) \frac{y_{i-} \mu_i}{\widehat{\sigma}} \right], r = 1, 2, \ldots, p, \\
\widehat{\mathbf{J}}_{r,p+2} &= - \sum_{i=1}^{n} \frac{\widehat{A}}{\widehat{B}} \frac{x_{ir}}{\widehat{\sigma}}, r = 1, 2, \ldots, p,
\end{aligned}
$$

and

$$
\begin{aligned}
\widehat{\mathbf{J}}_{p+1,p+1} &= \sum_{i=1}^{n} \frac{\widehat{A}(L - \widehat{\mu}_i)}{\widehat{\sigma}^3} \left[\frac{\widehat{A}}{\widehat{B}} \frac{L - \widehat{\mu}_i}{\widehat{\sigma}} + \frac{(L - \widehat{\mu}_i)^2}{\widehat{\sigma}^2} - 2 \right] - (1 - B) \frac{1}{\widehat{\sigma}^2} \left[1 - 3 \frac{(y_{i-} \widehat{\mu}_i)^2}{\widehat{\sigma}^2} \right], \\
\widehat{\mathbf{J}}_{p+1,p+2} &= - \sum_{i=1}^{n} \frac{\widehat{A}}{\widehat{B}} \frac{L - \widehat{\mu}_i}{\widehat{\sigma}^2}, \\
\widehat{\mathbf{J}}_{p+2,p+2} &= \sum_{i=1}^{n} \frac{\left(1 - 2\widehat{B}\right) + \widehat{B}}{\widehat{B}} = \sum_{i=1}^{n} \frac{1 - \widehat{B}}{\widehat{B}}.
\end{aligned}
$$

Partitioning $\mathbf{J}(\widehat{\beta}, \widehat{\sigma}, 0)$ as $\begin{bmatrix} \widehat{\mathbf{J}}_{11} & \widehat{\mathbf{J}}_{12} & \widehat{\mathbf{J}}_{13} \\ \widehat{\mathbf{J}}_{21} & \widehat{\mathbf{J}}_{22} & \widehat{\mathbf{J}}_{23} \\ \widehat{\mathbf{J}}_{31} & \widehat{\mathbf{J}}_{32} & \widehat{\mathbf{J}}_{33} \end{bmatrix}$, and let $\mathbf{L}$ be a $n \times 1$ vector having a 1 as all the elements, so we have

$$
\begin{aligned}
\widehat{\mathbf{J}}_{11} &= \mathbf{X}^T diag\left[\frac{1}{\widehat{\sigma}^2}\left(\frac{\widehat{A}_i^2}{\widehat{B}_i} + \widehat{A}_i\frac{(L-\widehat{\mu}_i)}{\widehat{\sigma}} + 1 - B\right)\right]\mathbf{X}, \\
\widehat{\mathbf{J}}_{12} &= \mathbf{X}^T diag\left[\frac{1}{\widehat{\sigma}^2}\left(\frac{\widehat{A}_i^2}{\widehat{B}_i}\frac{L-\widehat{\mu}_i}{\widehat{\sigma}} + \widehat{A}_i\frac{(L-\widehat{\mu}_i)^2}{\widehat{\sigma}^2} - \widehat{A}_i\right) + 2\,(1-B)\,\frac{y_{i-}\mu_i}{\widehat{\sigma}}\right]\mathbf{L}, \\
\widehat{\mathbf{J}}_{13} &= -\mathbf{X}^T diag\left[\frac{1}{\widehat{\sigma}}\frac{\widehat{A}_i}{\widehat{B}_i}\right]\mathbf{L}\,, \\
\widehat{\mathbf{J}}_{22} &= \sum_{i=1}^{n}\frac{\widehat{A}_i(L-\widehat{\mu}_i)}{\widehat{\sigma}^3}\left[\frac{\widehat{A}_i}{\widehat{B}_i}\frac{L-\widehat{\mu}_i}{\widehat{\sigma}} + \frac{(L-\widehat{\mu}_i)^2}{\widehat{\sigma}^2} - 2\right] - (1-B)\,\frac{1}{\widehat{\sigma}^2}\left[1 - 3\frac{(y_{i-}\widehat{\mu}_i)^2}{\widehat{\sigma}^2}\right], \\
\widehat{\mathbf{J}}_{23} &= -\sum_{i=1}^{n}\frac{\widehat{A}_i}{\widehat{B}_i}\frac{L-\widehat{\mu}_i}{\widehat{\sigma}^2}\ , \\
\widehat{\mathbf{J}}_{33} &= \sum_{i=1}^{n}\frac{1-\widehat{B}_i}{\widehat{B}_i},
\end{aligned}
$$

where $\widehat{\mathbf{J}}_{11}$ is a $p \times p$ matrix, $\widehat{\mathbf{J}}_{12}$ and $\widehat{\mathbf{J}}_{13}$ are $p \times 1$ matrix, $\widehat{\mathbf{J}}_{22}, \widehat{\mathbf{J}}_{23}$ and $\widehat{\mathbf{J}}_{33}$ are scalar. Other cells $\widehat{\mathbf{J}}_{21} = \widehat{\mathbf{J}}_{12}^T$, $\widehat{\mathbf{J}}_{31} = \widehat{\mathbf{J}}_{13}^T$ and $\widehat{\mathbf{J}}_{32} = \widehat{\mathbf{J}}_{23}^T$.

We define $\mathbf{J}(\widehat{\beta}, \widehat{\sigma}, 0) = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix}$, where $D_{11} = \begin{bmatrix} \widehat{\mathbf{J}}_{11} & \widehat{\mathbf{J}}_{12} \\ \widehat{\mathbf{J}}_{21} & \widehat{\mathbf{J}}_{22} \end{bmatrix}$, $D_{12} = \begin{bmatrix} \widehat{\mathbf{J}}_{13} \\ \widehat{\mathbf{J}}_{23} \end{bmatrix}$, $D_{21} = \begin{bmatrix} \widehat{\mathbf{J}}_{31} & \widehat{\mathbf{J}}_{32} \end{bmatrix}$ and $D_{22} = \widehat{\mathbf{J}}_{33}$. We also denote the inverse of $\mathbf{J}(\widehat{\beta}, \widehat{\sigma}, 0)$, $\left[\mathbf{J}(\widehat{\beta}, \widehat{\sigma}, 0)\right]^{-1}$, as C can be partitioned as $\left[\mathbf{J}(\widehat{\beta}, \widehat{\sigma}, 0)\right]^{-1} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$. Due to the structure of $\mathbf{J}(\widehat{\beta}, \widehat{\sigma}, 0)$, only C_{22} is needed and can be expressed as

$$
C_{22}^{-1} = D_{22} - D_{21}D_{11}^{-1}D_{12}. \tag{S.11}
$$

Based on a blockwise inversion formula, we have

$$D_{11}^{-1} = \begin{bmatrix} \widehat{\mathbf{J}}_{11} & \widehat{\mathbf{J}}_{12} \\ \widehat{\mathbf{J}}_{21} & \widehat{\mathbf{J}}_{22} \end{bmatrix}^{-1} = \begin{bmatrix} \widehat{\mathbf{J}}_{11}^{-1} + \widehat{\mathbf{J}}_{11}^{-1}\widehat{\mathbf{J}}_{12}\left(\widehat{\mathbf{J}}_{22} - \widehat{\mathbf{J}}_{21}\widehat{\mathbf{J}}_{11}^{-1}\widehat{\mathbf{J}}_{12}\right)^{-1}\widehat{\mathbf{J}}_{21}\widehat{\mathbf{J}}_{11}^{-1} & -\widehat{\mathbf{J}}_{11}^{-1}\widehat{\mathbf{J}}_{12}\left(\widehat{\mathbf{J}}_{22} - \widehat{\mathbf{J}}_{21}\widehat{\mathbf{J}}_{11}^{-1}\widehat{\mathbf{J}}_{12}\right)^{-1} \\ -\left(\widehat{\mathbf{J}}_{22} - \widehat{\mathbf{J}}_{21}\widehat{\mathbf{J}}_{11}^{-1}\widehat{\mathbf{J}}_{12}\right)^{-1}\widehat{\mathbf{J}}_{21}\widehat{\mathbf{J}}_{11}^{-1} & \left(\widehat{\mathbf{J}}_{22} - \widehat{\mathbf{J}}_{21}\widehat{\mathbf{J}}_{11}^{-1}\widehat{\mathbf{J}}_{12}\right)^{-1} \end{bmatrix}. \tag{S.12}$$

Plug (S.12) into (S.11), and let $V = \widehat{\mathbf{J}}_{11}^{-1}\widehat{\mathbf{J}}_{12}$ and $W = \left(\widehat{\mathbf{J}}_{22} - \widehat{\mathbf{J}}_{21}\widehat{\mathbf{J}}_{11}^{-1}\widehat{\mathbf{J}}_{12}\right)^{-1}$, we have

$$\begin{aligned}
C_{22}^{-1} &= \widehat{\mathbf{J}}_{33} - \begin{bmatrix} \widehat{\mathbf{J}}_{31} & \widehat{\mathbf{J}}_{32} \end{bmatrix} \begin{bmatrix} \widehat{\mathbf{J}}_{11}^{-1} + VWV^T & -VW \\ -WV^T & W \end{bmatrix} \begin{bmatrix} \widehat{\mathbf{J}}_{13} \\ \widehat{\mathbf{J}}_{23} \end{bmatrix} \\
&= \widehat{\mathbf{J}}_{33} - \begin{bmatrix} \widehat{\mathbf{J}}_{31}\widehat{\mathbf{J}}_{11}^{-1} + \widehat{\mathbf{J}}_{31}VWV^T - \widehat{\mathbf{J}}_{32}WV^T & -\widehat{\mathbf{J}}_{31}VW + \widehat{\mathbf{J}}_{32}W \end{bmatrix} \begin{bmatrix} \widehat{\mathbf{J}}_{13} \\ \widehat{\mathbf{J}}_{23} \end{bmatrix} \\
&= \widehat{\mathbf{J}}_{33} - \widehat{\mathbf{J}}_{31}\widehat{\mathbf{J}}_{11}^{-1}\widehat{\mathbf{J}}_{13} + \widehat{\mathbf{J}}_{31}VWV^T\widehat{\mathbf{J}}_{13} - \widehat{\mathbf{J}}_{32}WV^T\widehat{\mathbf{J}}_{13} - \widehat{\mathbf{J}}_{31}VW\widehat{\mathbf{J}}_{23} + \widehat{\mathbf{J}}_{32}W\widehat{\mathbf{J}}_{23} \\
&= \widehat{\mathbf{J}}_{33} - \widehat{\mathbf{J}}_{31}\widehat{\mathbf{J}}_{11}^{-1}\widehat{\mathbf{J}}_{13} - \widehat{\mathbf{J}}_{31}VWV^T\widehat{\mathbf{J}}_{13} + \widehat{\mathbf{J}}_{32}WV^T\widehat{\mathbf{J}}_{13} + \widehat{\mathbf{J}}_{31}VW\widehat{\mathbf{J}}_{23} - \widehat{\mathbf{J}}_{32}W\widehat{\mathbf{J}}_{23} \\
&= \widehat{\mathbf{J}}_{33} - \widehat{\mathbf{J}}_{31}\widehat{\mathbf{J}}_{11}^{-1}\widehat{\mathbf{J}}_{13} - \widehat{\mathbf{J}}_{31}VWV^T\widehat{\mathbf{J}}_{13} + \widehat{\mathbf{J}}_{32}WV^T\widehat{\mathbf{J}}_{13} + \widehat{\mathbf{J}}_{31}VW\widehat{\mathbf{J}}_{23} - \widehat{\mathbf{J}}_{32}W\widehat{\mathbf{J}}_{23}.
\end{aligned} \tag{S.13}$$

So the score statistic for testing $\omega = 0$ can be written as

$$\begin{aligned}
\mathbf{S}(\widehat{\beta}, \widehat{\sigma}) &= \mathbf{S}(\widehat{\beta}, \widehat{\sigma}, 0) = U^T(\widehat{\beta}, \widehat{\sigma}, 0)\left[\mathbf{J}(\widehat{\beta}, \widehat{\sigma}, 0)\right]^{-1} U(\widehat{\beta}, \widehat{\sigma}, 0) \\
&= \frac{\sum_{i=1}^n \left\{\frac{r_i - \widehat{B}_i}{\widehat{B}_i}\right\}}{\widehat{\mathbf{J}}_{33} - \widehat{\mathbf{J}}_{31}\widehat{\mathbf{J}}_{11}^{-1}\widehat{\mathbf{J}}_{13} - \widehat{\mathbf{J}}_{31}VWV^T\widehat{\mathbf{J}}_{13} + \widehat{\mathbf{J}}_{32}WV^T\widehat{\mathbf{J}}_{13} + \widehat{\mathbf{J}}_{31}VW\widehat{\mathbf{J}}_{23} - \widehat{\mathbf{J}}_{32}W\widehat{\mathbf{J}}_{23}}.
\end{aligned} \tag{S.14}$$

Under the null hypothesis, the statistic $\mathbf{S}(\widehat{\beta}, \widehat{\sigma})$ has an asymptotic chi-square distribution with one degree of freedom.

Table S1: The rejection rate for testing $H_0 : \omega = 0$ vs $H_A : \omega > 0$ for the Tobit model with no covariate.

Sample	$\mu = -1.5$			$\mu = -0.5$			$\mu = 0.5$			$\mu = 1.5$		
Size	Wald	LR	Score	Wald	LR	Score	Wald	LR	Score	Wald	LR	Score
50	0.333	0.087	0.177	0.231	0.070	0.132	0.148	0.076	0.111	0.096	0.063	0.087
100	0.246	0.056	0.108	0.203	0.065	0.113	0.129	0.057	0.086	0.084	0.054	0.068
200	0.218	0.083	0.114	0.138	0.051	0.072	0.115	0.065	0.079	0.089	0.061	0.072
500	0.148	0.062	0.082	0.123	0.062	0.073	0.101	0.064	0.073	0.079	0.056	0.062
1000	0.143	0.066	0.081	0.093	0.048	0.056	0.090	0.059	0.067	0.062	0.048	0.051

Table S2: The rejection rate for testing $H_0 : \omega = 0$ vs $H_A : \omega > 0$ for the Tobit model with a uniformly distributed covariate.

Sample	$\alpha = -1.0$			$\alpha = 0.0$			$\alpha = 1.0$			$\alpha = 2.0$		
Size	Wald	LR	Score	Wald	LR	Score	Wald	LR	Score	Wald	LR	Score
50	0.325	0.101	0.169	0.249	0.097	0.124	0.166	0.084	0.097	0.124	0.093	0.096
100	0.285	0.084	0.130	0.167	0.065	0.089	0.142	0.054	0.083	0.086	0.063	0.073
200	0.205	0.063	0.098	0.151	0.055	0.074	0.107	0.056	0.073	0.075	0.054	0.056
500	0.152	0.071	0.080	0.126	0.064	0.075	0.075	0.046	0.051	0.070	0.057	0.060
1000	0.142	0.068	0.078	0.086	0.049	0.054	0.080	0.058	0.063	0.059	0.049	0.052

Table S3: The rejection rate for testing $H_0 : \omega = 0$ vs $H_A : \omega > 0$ for the Tobit model with a normally distributed covariate.

Sample	$\alpha = -1.5$			$\alpha = -0.50$			$\alpha = 0.5$			$\alpha = 1.5$		
Size	Wald	LR	Score	Wald	LR	Score	Wald	LR	Score	Wald	LR	Score
50	0.213	0.085	0.126	0.132	0.071	0.086	0.091	0.064	0.057	0.047	0.073	0.045
100	0.159	0.062	0.088	0.107	0.064	0.070	0.061	0.046	0.042	0.032	0.049	0.029
200	0.134	0.072	0.083	0.083	0.055	0.055	0.067	0.063	0.047	0.038	0.054	0.023
500	0.107	0.059	0.066	0.069	0.048	0.039	0.052	0.048	0.035	0.031	0.045	0.022
1000	0.087	0.054	0.056	0.053	0.039	0.034	0.054	0.056	0.039	0.034	0.045	0.022

Table S4: The mean of estimated ω with varying percentage of censored data

	Prevalence	Percentage Under Detection						
Covariate	(ω)	60%	40%	20%	10%	6%	4%	2%
No		-0.0828	-0.0196	-0.0002	0.0003	-0.0008	-0.0002	-0.0001
Uniform	$\omega = 0$	-0.0740	-0.0098	-0.0038	-0.0016	-0.0002	-0.0001	-0.0003
Normal		-0.0183	-0.0094	-0.0038	-0.0022	-0.0024	-0.0023	-0.0022
No		-0.0477	0.0863	0.0969	0.0992	0.0998	0.0997	0.0998
Uniform	$\omega = 10\%$	0.0279	0.0878	0.1006	0.0988	0.0995	0.1000	0.1006
Normal		0.0921	0.0948	0.0991	0.1005	0.0995	0.1003	0.0995
No		0.1034	0.1819	0.1971	0.1991	0.1999	0.1997	0.1996
Uniform	$\omega = 20\%$	0.1129	0.1873	0.1954	0.1996	0.1991	0.2001	0.1999
Normal		0.1843	0.1937	0.1989	0.1993	0.2003	0.1996	0.1994
No		0.2160	0.2905	0.2990	0.2993	0.3000	0.2997	0.3004
Uniform	$\omega = 30\%$	0.2135	0.2868	0.2984	0.2990	0.3000	0.3004	0.2994
Normal		0.2874	0.2970	0.2993	0.2989	0.2984	0.2999	0.3003

Table S5: Rejection rate or power (%) for testing the latent class with varying prevalence of the latent class and percentage of censored data for the multivariates Tobit model (??)

Prevalence (ω)	Statistical Test	Percentage Under Detection 10%	20%	30%	40%	50%	60%
$\omega = 0.0^{\star}$	Wald test	4.69	5.37	4.89	5.20	5.69	5.85
	LR test	3.41	4.21	3.22	3.23	3.11	2.97
	Score test	4.80	5.10	5.90	6.00	5.90	6.60
$\omega = 10\%$	Wald test	100.00	100.00	98.80	83.60	53.50	33.60
	LR test	90.60	90.60	90.10	80.60	52.80	30.90
	Score test	100.00	100.00	99.50	89.70	61.30	38.00
$\omega = 20\%$	Wald test	100.00	100.00	100.00	98.50	92.00	63.80
	LR test	99.20	99.20	99.20	99.10	93.50	64.00
	Score test	100.00	100.00	100.00	100.00	97.40	75.20
$\omega = 30\%$	Wald test	100.00	100.00	100.00	99.50	94.08	74.19
	LR test	100.00	100.00	100.00	100.00	99.00	79.40
	Score test	100.00	100.00	100.00	100.00	99.80	94.40

Note: $\star$ for the rejection rates. The true type I error is 5%.