

Appendix. Thermodynamic constitutive model for the SMA

A.1 Gibbs free energy of SMA materials

SMA materials generally have two phases, that are, austenite phase and martensite phase. It is assumed that the volume fraction of each phase is ξ^α , $\alpha = M \text{ or } A$, (superscript “ M ” refers to the martensitic phase and superscript “ A ” refers to the austenitic phase, respectively), and $\xi^A + \xi^M = 1$. Set $\xi^M = \xi$, then $\xi^A = 1 - \xi$. According to the research of Boyd and Lagoudas (1996), the total specific Gibbs free energy, G , of SMA materials is assumed to be equal to the mass weighted sum of the free energy G^α , of each phase plus the free energy G^{mix} , of mixing,

$$\begin{aligned} G(\sigma_{ij}, T, \xi, \varepsilon_{ij}^t) &= \xi^\alpha G^\alpha(\sigma_{ij}, T) + G^{mix}(\sigma_{ij}, T, \xi, \varepsilon_{ij}^t) \\ &= (1 - \xi) G^A(\sigma_{ij}, T) + \xi G^M(\sigma_{ij}, T) + G^{mix}(\sigma_{ij}, T, \xi, \varepsilon_{ij}^t) \end{aligned} \quad (A1)$$

where G^A is the free energy of austenite phase, G^M is the free energy of martensite phase, ξ is the volume fraction of martensite phase, T is temperature, σ_{ij} is stress tensor, ε_{ij}^t is transformation strain tensor.

The free energy of each phase can be expressed as (Boyd and Lagoudas, 1996),

$$\begin{aligned} G^\alpha(\sigma_{ij}, T) &= -\frac{1}{2} \frac{1}{\rho^\alpha} S_{ijkl}^\alpha \sigma_{ij} \sigma_{kl} - \frac{1}{\rho^\alpha} \alpha_{ij}^\alpha \sigma_{ij} (T - T_0) + c^\alpha \left[(T - T_0) - T \ln \left(\frac{T}{T_0} \right) \right] \\ &\quad - s_0^\alpha T + u_0^\alpha \end{aligned} \quad (A2)$$

and the free energy of mixing is assumed as (Lagoudas et al., 1995),

$$G^{mix}(\sigma_{ij}, T, \xi, \varepsilon_{ij}^t) = \frac{1}{\rho} f(\xi, \varepsilon_{ij}^t) \quad (A3)$$

where ρ^α , S_{ijkl}^α , α_{ij}^α , c^α , s_0^α , u_0^α are mass density, elastic compliance tensor, thermal expansion coefficient tensor, specific heat, specific entropy at a reference state, and specific internal energy at a reference state of the “ α ” phase, respectively. T_0 is the temperature at a reference state. The generic function $f(\xi, \varepsilon_{ij}^t)$ physically represents the elastic strain energy due to the interaction between martensitic variants and the surrounding austenite phase, and among the martensitic variants themselves, the detailed derivation will be shown in Section

A.2. Reorientation (detwinning) effects have been omitted for simplicity (Lagoudas et al., 1995).

According to the second law of thermodynamics outlined by Coleman and Noll (1963), the local internal dissipation rate can be expressed as,

$$T \dot{\eta} = -\rho \dot{G} - \rho s \dot{T} - \varepsilon_{ij}^{te} \dot{\sigma}_{ij} + \sigma_{ij} \dot{\varepsilon}_{ij}^t \geq 0 \quad (\text{A4})$$

where $\dot{\eta}$ is the local dissipation rate without the entropy production rate due to heat conduction, the free energy rate \dot{G} can be expressed as equation (A5) (Boyd and Lagoudas, 1996) and the “dot” above is a symbol that indicates the increment of the corresponding quantity,

$$\dot{G} = \frac{\partial G}{\partial \sigma_{ij}} \dot{\sigma}_{ij} + \frac{\partial G}{\partial T} \dot{T} + \frac{\partial G}{\partial \xi} \dot{\xi} + \frac{\partial G}{\partial \varepsilon_{ij}^t} \dot{\varepsilon}_{ij}^t \quad (\text{A5})$$

Using equations (A1), (A2), (A3) and (A5), then the inequality (A4) can be rewritten as,

$$T \dot{\eta} = - \left(\varepsilon_{ij}^{te} + \rho \frac{\partial G}{\partial \sigma_{ij}} \right) \dot{\sigma}_{ij} - \rho \left(\frac{\partial G}{\partial T} + s \right) \dot{T} + \left(\sigma_{ij} - \frac{\partial f}{\partial \varepsilon_{ij}^t} \right) \dot{\varepsilon}_{ij}^t - \rho \frac{\partial G}{\partial \xi} \dot{\xi} \geq 0 \quad (\text{A6})$$

where $\varepsilon_{ij}^{te} = \varepsilon_{ij} - \varepsilon_{ij}^t$ is the thermoelastic strain tensor, ε_{ij} is the total strain tensor, ρ is the mass density of the SMA and s is the entropy per unit mass.

Since σ_{ij} and T are independent state variables, and G is independent of $\dot{\sigma}_{ij}$ and \dot{T} , then the thermoelastic strain tensor ε_{ij}^{te} and the entropy per unit mass s can be defined as,

$$\varepsilon_{ij}^{te} = \varepsilon_{ij} - \varepsilon_{ij}^t = -\rho \frac{\partial G}{\partial \sigma_{ij}} = S_{ijkl} \sigma_{kl} + \alpha_{ij} (T - T_0) \quad (\text{A7})$$

$$s = -\frac{\partial G}{\partial T} = \frac{1}{\rho} \alpha_{ij} \sigma_{ij} + c \ln \left(\frac{T}{T_0} \right) + s_0 \quad (\text{A8})$$

where $S_{ijkl} = S_{ijkl}^A + \xi (S_{ijkl}^M - S_{ijkl}^A)$, $\alpha_{ij} = \alpha_{ij}^A + \xi (\alpha_{ij}^M - \alpha_{ij}^A)$, $c_{ij} = c^A + \xi (c^M - c^A)$,

$$s_0 = s_0^A + \xi (s_0^M - s_0^A).$$

Using equations (A7) and (A8), the inequality (A6) can be simplified as,

$$T \dot{\eta} = \left(\sigma_{ij} - \frac{\partial f}{\partial \varepsilon_{ij}^t} \right) \dot{\varepsilon}_{ij}^t - \rho \frac{\partial G}{\partial \xi} \dot{\xi} \geq 0 \quad (\text{A9})$$

and the effective stress σ_{ij}^{eff} , acting as a thermodynamic force conjugate to ε_{ij}^t , can be expressed as (Lagoudas et al., 1995),

$$\sigma_{ij}^{eff} = \sigma_{ij} - \frac{\partial f}{\partial \varepsilon_{ij}^t} \quad (\text{A10})$$

then the inequality (A9) can be simplified as,

$$T \dot{\eta} = \sigma_{ij}^{eff} \dot{\varepsilon}_{ij}^t - \rho \frac{\partial G}{\partial \xi} \dot{\xi} \geq 0 \quad (\text{A11})$$

In this work, only transformation is considered, while reorientation of martensitic variants is disregarded, so the following assumptions can be introduced to simplify the formulation given above (Bondaryev and Wayman, 1988; Boyd and Lagoudas, 1996),

$$\dot{\varepsilon}_{ij}^t = \Lambda_{ij} \dot{\xi} \quad (\text{A12})$$

$$\Lambda_{ij} = \begin{cases} \frac{3}{2} H (\bar{\sigma}^{eff})^{-1} \sigma_{ij}^{eff} & , \dot{\xi} > 0 \\ H (\bar{\varepsilon}^t)^{-1} \varepsilon_{ij}^t & , \dot{\xi} < 0 \end{cases} \quad (\text{A13})$$

where Λ_{ij} is the transformation tensor, H is the maximum uniaxial transformation strain,

$$\bar{\sigma}^{eff} = \left(\frac{3}{2} \sigma_{ij}^{eff} \sigma_{ij}^{eff} \right)^{\frac{1}{2}}, \quad \sigma_{ij}^{eff} = \sigma_{ij}^{eff} - \frac{1}{3} \sigma_{kk}^{eff} \delta_{ij} \quad \text{and} \quad \bar{\varepsilon}^t = \left(\frac{2}{3} \varepsilon_{ij}^t \varepsilon_{ij}^t \right)^{\frac{1}{2}}.$$

Using equation (A12), the inequality (A11) can be rewritten as,

$$T \dot{\eta} = \left(\sigma_{ij}^{eff} \Lambda_{ij} - \rho \frac{\partial G}{\partial \xi} \right) \dot{\xi} = \Pi \dot{\xi} \geq 0 \quad (\text{A14})$$

where Π is the thermodynamic force conjugate to ξ , and can be expressed as,

$$\Pi = \sigma_{ij}^{eff} \Lambda_{ij} - \rho \frac{\partial G}{\partial \xi} = \sigma_{ij}^{eff} \Lambda_{ij} + \frac{1}{2} \Delta S_{ijkl} \sigma_{ij} \sigma_{kl} + \Delta \alpha_{ij} \sigma_{ij} \Delta T - \rho \Delta c \left[\Delta T - T \ln \left(\frac{T}{T_0} \right) \right] +$$

$$\rho\Delta s_0 T - \rho\Delta u_0 - \frac{\partial f}{\partial \xi} \quad (\text{A15})$$

and $\Delta S_{ijkl} = S_{ijkl}^M - S_{ijkl}^A$, $\Delta\alpha_{ij} = \alpha_{ij}^M - \alpha_{ij}^A$, $\Delta T = T - T_0$, $\Delta c = c^M - c^A$, $\Delta s_0 = s_0^M - s_0^A$, $\Delta u_0 = u_0^M - u_0^A$.

According to the inequality (A14), for the forward phase transformation (when $\dot{\xi} > 0$) the thermodynamic force Π must be greater than zero, and for the reverse phase transformation (when $\dot{\xi} < 0$) Π must be less than zero.

A.2 Martensitic volume fraction

To obtain the evolution equation for the martensitic volume fraction ξ , a standard formalism of thermodynamic dissipation potentials is utilized (Edelen, 1974). Following Edelen's formalism, a dissipation potential, $\varphi(\Pi; \sigma_{ij}, T, \xi, \varepsilon_{ij}^t)$, can be introduced such that the evolution equation of the internal state variable, ξ , for the rate independent case, is given by (Lagoudas et al., 1995),

$$\dot{\xi} = \lambda \frac{\partial \varphi(\Pi; \sigma_{ij}, T, \xi, \varepsilon_{ij}^t)}{\partial \Pi} \quad (\text{A16})$$

where λ is the lame constant and should satisfy the Kuhn-Tucker conditions,

$$\lambda \geq 0, \quad \varphi \leq Y, \quad \lambda(\varphi - Y) = 0 \quad (\text{A17})$$

and Y is a material parameter related to the dissipation rate of the system, and is assumed to be constant during phase transformation. A convex quadratic functional representation of the dissipation potential was assumed by Lagoudas et al. (1995),

$$\varphi = \frac{1}{2} \Pi^2 \quad (\text{A18})$$

During the phase transformation, λ is always greater than zero. To satisfy the Kuhn-Tucker conditions, φ must be equal to Y , then the following equation can be obtained by using equation (A18),

$$\Pi = \pm\sqrt{2\varphi} = \pm\sqrt{2Y} = \pm Y^*; \quad Y^* = \sqrt{2Y} \quad (\text{A19})$$

where $Y^* = \sqrt{2Y}$ can be interpreted as the threshold value of the thermodynamic force Π for the onset of the

phase transformation. According to the inequality (A14),

$$\Pi = \begin{cases} Y^* & , \quad \dot{\xi} > 0 \\ -Y^* & , \quad \dot{\xi} < 0 \end{cases} \quad (\text{A20})$$

In order to get the function $f(\xi, \varepsilon_{ij}^t)$, the following assumptions were put forwarded by Lagoudas et al. (1995),

- (1) f is only dependent on ξ and is independent of ε_{ij}^t , which means, $G^{mix} = \frac{f(\xi)}{\rho}$,
- (2) the SMA is stress free if no external mechanical loading is applied at the fully austenitic state, which means, $f(0) = 0$,
- (3) the function $f(\xi)$ must be non-negative, which means $f(\xi) \geq 0$, $0 \leq \xi \leq 1$,
- (4) the function $f(\xi)$ must be continuous during the whole phase transformation.

Applying the above constraints, a form for the function $f(\xi)$ can be selected as follows,

$$f(\xi) = \begin{cases} f^M(\xi) & , \quad \dot{\xi} > 0 \\ f^A(\xi) & , \quad \dot{\xi} < 0 \end{cases} \quad (\text{A21})$$

where

$$f^M(\xi) = f^{M0}(\xi) + \frac{1-\xi}{1-\xi^R} [f^A(\xi^R) - f^{M0}(\xi^R)] \quad (\text{A22})$$

$$f^A(\xi) = f^{A0}(\xi) + \frac{\xi}{\xi^R} [f^M(\xi^R) - f^{A0}(\xi^R)] \quad (\text{A23})$$

and ξ^R is the martensitic volume fraction at the return point (a return point during the phase transformation is characterized by a change in the sign of $\dot{\xi}$). For the forward phase transformation, $\xi^R \leq \xi \leq 1$, while $0 \leq \xi \leq \xi^R$ for the reverse phase transformation.

According to the research of Liang and Rogers (1990), functions $f^{M0}(\xi)$ and $f^{A0}(\xi)$ can be expressed as,

$$\begin{cases} f^{M0}(\xi) = \int_0^\xi -\frac{\rho\Delta s_0}{a^M} [\pi - \cos^{-1}(2\xi - 1)] d\xi + (u_1 + u_2)\xi \\ f^{A0}(\xi) = \int_0^\xi -\frac{\rho\Delta s_0}{a^A} [\pi - \cos^{-1}(2\xi - 1)] d\xi + (u_1 - u_2)\xi \end{cases} \quad (A24)$$

where $a^M = \frac{\pi}{M^s - M^f}$, $a^A = \frac{\pi}{A^f - A^s}$, $u_1 = \frac{1}{2} \rho\Delta s_0 (M^s + A^f) - \rho\Delta u_0$, $u_2 = \frac{\pi\rho\Delta s_0}{4} \left(\frac{1}{a^M} - \frac{1}{a^A} \right)$,

M^s is the martensitic transformation start temperature, M^f is the martensitic transformation finish temperature,

A^s is the austenitic transformation start temperature, A^f is the austenitic transformation finish temperature.

According to equation (A20), for the forward phase transformation ($\dot{\xi} > 0$), $\Pi = Y^*$. Using equations (A15), (A21), (A22), (A23) and (A24), the following equation can be obtained,

$$\begin{aligned} \sigma_{ij}^{eff} \Lambda_{ij} + \frac{1}{2} \Delta S_{ijkl} \sigma_{ij} \sigma_{kl} + \Delta \alpha_{ij} \sigma_{ij} \Delta T - \rho \Delta c \left[\Delta T - T \ln \left(\frac{T}{T_0} \right) \right] + \rho \Delta s_0 T - \rho \Delta u_0 - \\ \frac{\rho \Delta s_0}{a^M} [\cos^{-1}(2\xi - 1) - \pi] - (u_1 + u_2) + \frac{f^A(\xi^R) - f^{M0}(\xi^R)}{1 - \xi^R} - Y^* = 0 \end{aligned} \quad (A25)$$

where $Y^* = -\frac{1}{2} \rho \Delta s_0 (A^f - M^s) - \frac{1}{4} \rho \Delta s_0 (M^s - M^f - A^f + A^s)$ (Boyd and lagoudas, 1996). For the case

of a complete loading-unloading cycle, $\xi^R = 0$, which means $f^A(\xi^R) = f^{M0}(\xi^R) = 0$. And $\Delta c = 0$ is assumed. After arrangement of equation (A25), the expresstion of ξ for the forward phase transformation can be obtained,

$$\xi = \frac{1}{2} \left\{ \cos \left[a^M (T - M^f) - \frac{a^M}{C^M H} \left(\sigma_{ij} \Lambda_{ij} + \frac{1}{2} \Delta S_{ijkl} \sigma_{ij} \sigma_{kl} + \Delta \alpha_{ij} \sigma_{ij} \Delta T \right) \right] + 1 \right\}, \quad \dot{\xi} > 0 \quad (A26)$$

where the constant $C^M = -\frac{\rho \Delta s_0}{H}$ is the martensite stress influence coefficient.

Similarly, using equations (A15), (A20), (A21), (A22), (A23) and (A24), the expresstion of ξ for the reverse phase transformation can be obtained,

$$\xi = \frac{1}{2} \left\{ \cos \left[a^A (T - A^s) - \frac{a^A}{C^M H} \left(\sigma_{ij} \Lambda_{ij} + \frac{1}{2} \Delta S_{ijkl} \sigma_{ij} \sigma_{kl} + \Delta \alpha_{ij} \sigma_{ij} \Delta T \right) \right] + 1 \right\} \quad , \quad \dot{\xi} < 0$$

(A27)