Appendix A. Elements of the matrix equation

The elements of the matrices in Eq. (57) are as follows:

$$a_{1,1} = h\left(-k_p^h\, H_{m+1}^1\left(k_p^h r\right) + \frac{m}{r} H_m^1\left(k_p^h r\right)\right) \tag{1}$$

$$a_{1,2} = h\frac{im}{r}\, H_m(k_s^h r) \tag{2}$$

$$a_{1,3} = (a-b)A_1^{>}\left(-k_1^h\, H_{m+1}^1(k_1^h r) + \frac{m}{r} H_m^1(k_1^h r)\right) \tag{3}$$

$$a_{1,4} = (a-b)A_2^{>}\left(-k_2^h\, H_{m+1}^1(k_2^h r) + \frac{m}{r} H_m^1(k_2^h r)\right) \tag{4}$$

$$a_{1,5} = (a-b)\frac{im}{r} H_m(k_3^h r) \tag{5}$$

$$a_{1,6} = -\left(h-(a+b)\right)\left(-k_p^{h-(a+b)}\, J_{m+1}\left(k_p^{h-(a+b)} r\right) + \frac{m}{r} J_m\left(k_p^{h-(a+b)} r\right)\right) \tag{6}$$

$$a_{1,7} = \left(h-(a+b)\right)\frac{im}{r} J_m\left(k_s^{h-(a+b)} r\right) \tag{7}$$

$$a_{1,8} = a_{1,9} = a_{1,10} = 0 \tag{8}$$

$$b_1 = -hi^m\left(-k_p\, J_{m+1}\left(k_p r\right) + \frac{m J_m\left(k_p r\right)}{r}\right) \tag{9}$$

$$a_{2,1} = \frac{im}{r} h\, H_m\left(k_p^h r\right) \tag{10}$$

$$a_{2,2} = h\left(k_s^h\, H_{m+1}^1(k_s^h r) - \frac{m}{r} H_m^1(k_s^h r)\right) \tag{11}$$

$$a_{2,3} = (a-b)\frac{imA_1^{>}}{r}\, H_m(k_1^h r) \tag{12}$$

$$a_{2,4} = (a-b)\frac{imA_2^{>}}{r} H_m(k_2^h r) \tag{13}$$

$$a_{2,5} = (a-b)\left(-k_3^h\, H_{m+1}^1(k_3^h r) + \frac{m}{r} H_m^1(k_3^h r)\right) \tag{14}$$

$$a_{2,6} = -\frac{im}{r}\left(h-(a+b)\right) J_m\left(k_p^{h-(a+b)} r\right) \tag{15}$$

$$a_{2,7} = \left(h-(a+b)\right)\left(k_s^{h-(a+b)}\, J_{m+1}\left(k_s^{h-(a+b)} r\right) - \frac{m}{r} J_m\left(k_s^{h-(a+b)} r\right)\right) \tag{16}$$

$$a_{2,8} = a_{2,9} = a_{2,10} = 0 \tag{17}$$

$$b_2 = -\frac{h}{r} m\, i^{m+1} J_m\left(k_p r\right) \tag{18}$$

$$a_{3,1} = a_{3,2} = a_{3,5} = a_{3,6} = a_{3,7} = a_{3,10} = 0 \tag{19}$$

$$a_{3,3} = H_m(k_1^h r) \tag{20}$$

$$a_{3,4} = H_m(k_2^h r) \tag{21}$$

$$a_{3,8} = -J_m\left(k_1^{h-(a+b)}r\right) \tag{22}$$

$$a_{3,9} = -J_m\left(k_2^{h-(a+b)}r\right) \tag{23}$$

$$b_3 = 0 \tag{24}$$

$$a_{4,1} = a_{4,2} = a_{4,6} = a_{4,7} = a_{4,9} = 0 \tag{25}$$

$$a_{4,3} = A_1^{>}\left(-k_1^h\, H_{m+1}^1(k_1^h r) + \frac{m}{r}H_m^1(k_1^h r)\right) \tag{26}$$

$$a_{4,4} = A_2^{>}\left(-k_2^h\, H_{m+1}^1(k_2^h r) + \frac{m}{r}H_m^1(k_2^h r)\right) \tag{27}$$

$$a_{4,5} = \frac{im}{r}H_m(k_3^h r) \tag{28}$$

$$\begin{aligned} a_{4,8} = -A_1^{<}&\left(-k_1^{h-(a+b)}\, J_{m+1}\left(k_1^{h-(a+b)}r\right) + \frac{mJ_m\left(k_1^{h-(a+b)}r\right)}{r}\right) \\ &- A_2^{<}\left(-k_2^{h-(a+b)}\, J_{m+1}\left(k_2^{h-(a+b)}r\right) + \frac{mJ_m\left(k_2^{h-(a+b)}r\right)}{r}\right) \end{aligned} \tag{29}$$

$$a_{4,10} = -\frac{im}{r}J_m\left(k_3^{h-(a+b)}r\right) \tag{30}$$

$$b_4 = 0 \tag{31}$$

$$a_{5,1} = a_{5,2} = a_{5,6} = a_{5,7} = 0 \tag{32}$$

$$a_{5,3} = \frac{imA_1^{>}}{r}\, H_m(k_1^h r) \tag{33}$$

$$a_{5,4} = \frac{imA_2^{>}}{r}H_m(k_2^h r) \tag{34}$$

$$a_{5,5} = -k_3^h\, H_{m+1}^1(k_3^h r) + \frac{m}{r}H_m^1(k_3^h r) \tag{35}$$

$$a_{5,8} = -\frac{imA_1^{<}}{r}\;\; J_m\left(k_1^{h-(a+b)}r\right) \tag{36}$$

$$a_{5,9} = -\frac{imA_2^{<}}{r}J_m\left(k_2^{h-(a+b)}r\right) \tag{37}$$

$$a_{5,10} = k_3^{h-(a+b)}\, J_{m+1}\left(k_3^{h-(a+b)}r\right) - \frac{mJ_m\left(k_3^{h-(a+b)}r\right)}{r} \tag{38}$$

$$b_5 = 0 \tag{39}$$

$$a_{6,1} = 2h\,\frac{Eh}{1-\nu^2}\left(\frac{(m^2-m)(1-\nu) - \left(k_p^h r\right)^2}{r^2}H_m^1\left(k_p^h r\right) + \frac{k_p^h}{r}(1-\nu)H_{m+1}^1\left(k_p^h r\right)\right) \tag{40}$$

$$a_{6,2} = 2\mathrm{h}\,\frac{E^2 h}{(1+\nu)^2(1-\nu)}\left(-\frac{imk_s^h}{r}\, H_{m+1}^1(k_s^h r) + \frac{i(m^2-m)}{r^2}\, H_m(k_s^h r)\right) \tag{41}$$

$$a_{6,3} = a_{6,4} = a_{6,5} = a_{6,8} = a_{6,9} = a_{6,10} = 0 \tag{42}$$

$$a_{6,6} = -2\big(h-(a+b)\big)^2 \frac{E}{1-\nu^2}\left(\frac{(m^2-m)(1-\nu)-\big(k_p^{h-(a+b)}r\big)^2}{r^2}J_m\big(k_p^{h-(a+b)}r\big) + \frac{k_p^{h-(a+b)}}{r}(1-\nu)J_{m+1}\big(k_p^{h-(a+b)}r\big)\right) \tag{43}$$

$$a_{6,7} = -2\big(h-(a+b)\big)^2 \frac{E^2}{(1+\nu)^2(1-\nu)}\left(-\frac{imk_s^{h-(a+b)}}{r}J_{m+1}\big(k_s^{h-(a+b)}r\big) + \frac{i(m^2-m)}{r^2}J_m\big(k_s^{h-(a+b)}r\big)\right) \tag{44}$$

$$b_6 = -2h\frac{Eh}{1-\nu^2}i^m\left(\frac{(m^2-m)(1-\nu)-\big(k_pr\big)^2}{r^2}J_m\big(k_pr\big)+\frac{k_p(1-\nu)}{r}J_{m+1}\big(k_pr\big)\right) \tag{45}$$

$$a_{7,1} = 2h\frac{Eh}{1+\nu}\left(-\frac{imk_p^h}{r}H_{m+1}^1\big(k_p^hr\big)+\frac{i(m^2-m)}{r^2}H_m\big(k_p^hr\big)\right) \tag{46}$$

$$a_{7,2} = -2h\frac{E^2}{2(1+\nu)^2}h\left(\frac{2m^2-2m-(k_s^hr)^2}{r^2}H_m^1(k_s^hr)+\frac{2k_s^h}{r}H_{m+1}^1(k_s^hr)\right) \tag{47}$$

$$a_{7,3} = a_{7,4} = a_{7,5} = a_{7,8} = a_{7,9} = a_{7,10} = 0 \tag{48}$$

$$a_{7,6} = -2\big(h-(a+b)\big)^2\frac{E}{1+\nu}\left(-\frac{imk_p^{h-(a+b)}}{r}J_{m+1}\big(k_p^{h-(a+b)}r\big)+\frac{i(m^2-m)}{r^2}J_m\big(k_p^{h-(a+b)}r\big)\right) \tag{49}$$

$$a_{7,7} = 2\big(h-(a+b)\big)^2\frac{E^2}{2(1+\nu)^2}\left(\frac{2m^2-2m-\big(k_s^{h-(a+b)}r\big)^2}{r^2}J_m\big(k_s^{h-(a+b)}r\big) + \frac{2k_s^{h-(a+b)}}{r}J_{m+1}\big(k_s^{h-(a+b)}r\big)\right) \tag{50}$$

$$b_7 = -2h\frac{Eh}{1+\nu}i^{m+1}\left(-\frac{mk_p}{r}J_{m+1}\big(k_pr\big)+\frac{m^2-m}{r^2}J_m\big(k_pr\big)\right) \tag{51}$$

$$a_{8,1} = a_{8,2} = 0 \tag{52}$$

$$a_{8,3} = \frac{8Eh^3}{12(1-\nu^2)}A_1^{\geq}\left(\left(\frac{m^2-m-(k_1^hr)^2}{r^2}H_m(k_1^hr)\right)+\frac{(1-\nu)k_1^h}{r}H_{m+1}^1(k_1^hr)\right) \tag{53}$$

$$a_{8,4} = \frac{8Eh^3}{12(1-\nu^2)}A_2^{\geq}\left(\frac{-(k_2^hr)^2+(m-m^2)(\nu-1)}{r^2}H_m(k_2^hr)+\frac{(1-\nu)k_2^h}{r}H_{m+1}^1(k_2^hr)\right) \tag{54}$$

$$a_{8,5} = \frac{8Eh^3}{12(1-\nu^2)}\left(\frac{im(\nu-1)+im^2(\nu+1)}{r^2}H_m^1(k_3^hr)-\frac{\nu+1}{r}imk_3^hH_{m+1}^1(k_3^hr)\right) \tag{55}$$

$$a_{8,6} = 2(a-b)\big(h-(a+b)\big)^2\frac{E}{1-\nu^2}\left(\frac{(m^2-m)(1-\nu)-\big(k_p^{h-(a+b)}r\big)^2}{r^2}J_m\big(k_p^{h-(a+b)}r\big) + \frac{k_p^{h-(a+b)}}{r}(1-\nu)J_{m+1}\big(k_p^{h-(a+b)}r\big)\right) \tag{56}$$

$$a_{8,7} = 2(a-b)\big(h-(a+b)\big)^2\frac{E^2}{(1+\nu)^2(1-\nu)}\left(-\frac{imk_s^{h-(a+b)}}{r}J_{m+1}\big(k_s^{h-(a+b)}r\big) + \frac{i(m^2-m)}{r^2}J_m\big(k_s^{h-(a+b)}r\big)\right) \tag{57}$$

$$a_{8,8} = -\frac{8E\left(h-(a+b)\right)^3}{12(1-\nu^2)} A_1^{<}\left(\frac{(m^2-m)(1-\nu)-\left(k_1^{h-(a+b)}r\right)^2}{r^2} J_m\left(k_1^{h-(a+b)}r\right) + \frac{k_1^{h-(a+b)}(1-\nu)}{r} J_{m+1}\left(k_1^{h-(a+b)}r\right)\right) \tag{58}$$

$$a_{8,9} = -\frac{8E\left(h-(a+b)\right)^3}{12(1-\nu^2)} A_2^{<}\left(\frac{(m^2-m)(1-\nu)-\left(k_2^{h-(a+b)}r\right)^2}{r^2} J_m\left(k_2^{h-(a+b)}r\right) + \frac{k_2^{h-(a+b)}(1-\nu)}{r} J_{m+1}\left(k_2^{h-(a+b)}r\right)\right) \tag{59}$$

$$a_{8,10} = -\frac{8E\left(h-(a+b)\right)^3}{12(1-\nu^2)}\left(\frac{im(\nu-1)+im^2(\nu+1)}{r^2} J_m\left(k_3^{h-(a+b)}r\right) - \frac{im(\nu+1)}{r} k_3^{h-(a+b)} J_{m+1}\left(k_3^{h-(a+b)}r\right)\right) \tag{60}$$

$$b_8 = 0 \tag{61}$$

$$a_{9,1} = a_{9,2} = 0 \tag{62}$$

$$a_{9,3} = \frac{8Eh^3}{24(1+\nu)} A_1^{>}\left(\frac{2}{r} im\left(-k_1^h H_{m+1}^1(k_1^h r)\right) + \frac{2i(m^2-m)}{r^2} H_m^1(k_1^h r)\right) \tag{63}$$

$$a_{9,4} = \frac{8Eh^3}{24(1+\nu)} A_2^{>}\left(\frac{2}{r} im\left(-k_2^h H_{m+1}^1(k_2^h r)\right) + \frac{2i(m^2-m)}{r^2} H_m(k_2^h r)\right) \tag{64}$$

$$a_{9,5} = \frac{8Eh^3}{24(1+\nu)}\left(\frac{-2m-(k_3^h r)^2}{r^2} H_m^1(k_3^h r) + \frac{2k_3^h}{r} H_{m+1}^1(k_3^h r)\right) \tag{65}$$

$$a_{9,6} = 2(a-b)\left(h-(a+b)\right)^2 \frac{E}{1+\nu}\left(-\frac{imk_p^{h-(a+b)}}{r} J_{m+1}\left(k_p^{h-(a+b)}r\right) + \frac{i(m^2-m)}{r^2} J_m\left(k_p^{h-(a+b)}r\right)\right) \tag{66}$$

$$a_{9,7} = -2(a-b)\left(h-(a+b)\right)^2 \frac{E^2}{2(1+\nu)^2}\left(\frac{2m^2-2m-\left(k_s^{h-(a+b)}r\right)^2}{r^2} J_m\left(k_s^{h-(a+b)}r\right) + \frac{2k_s^{h-(a+b)}}{r} J_{m+1}\left(k_s^{h-(a+b)}r\right)\right) \tag{67}$$

$$a_{9,8} = -\frac{8E\left(h-(a+b)\right)^3}{24(1+\nu)} A_1^{<}\left(-\frac{2imk_1^{h-(a+b)}}{r} J_{m+1}\left(k_1^{h-(a+b)}r\right) + \frac{2i(m^2-m)}{r^2} J_m\left(k_1^{h-(a+b)}r\right)\right) \tag{68}$$

$$a_{9,9} = -\frac{8E\left(h-(a+b)\right)^3}{24(1+\nu)} A_2^{<}\left(\frac{2i(m^2-m)}{r^2} J_m\left(k_2^{h-(a+b)}r\right) - \frac{2imk_2^{h-(a+b)}}{r} J_{m+1}\left(k_2^{h-(a+b)}r\right)\right) \tag{69}$$

$$a_{9,10} = -\frac{8E\left(h-(a+b)\right)^3}{24(1+\nu)}\left(\frac{-2m-\left(k_3^{h-(a+b)}r\right)^2}{r^2} J_m\left(k_3^{h-(a+b)}r\right) + \frac{2k_3^{h-(a+b)}}{r} J_{m+1}\left(k_3^{h-(a+b)}r\right)\right) \tag{70}$$

$$b_9 = 0 \tag{71}$$

$$a_{10,1} = a_{10,2} = a_{10,6} = a_{10,7} = 0 \tag{72}$$

$$a_{10,3} = \kappa^2 h\mu(A_1^{>}+1)\left(-k_1^h H_{m+1}^1(k_1^h r) + \frac{m}{r} H_m^1(k_1^h r)\right) \tag{73}$$

$$a_{10,4} = \kappa^2 h\mu(A_2^{>}+1)\left(-k_2^h H_{m+1}^1(k_2^h r) + \frac{m}{r} H_m^1(k_2^h r)\right) \tag{74}$$

$$a_{10,5} = \frac{1}{r} im\kappa^2 h\mu \, H_m(k_3^h r) \tag{75}$$

$$a_{10,8} = -\kappa^2\big(h-(a+b)\big)\mu(A_1^{<}+1)\left(-k_1^{h-(a+b)} J_{m+1}\big(k_1^{h-(a+b)}r\big) + \frac{m}{r} J_m\big(k_1^{h-(a+b)}r\big)\right) \tag{76}$$

$$a_{10,9} = -\kappa^2\big(h-(a+b)\big)\mu(A_2^{<}+1)\left(-k_2^{h-(a+b)} J_{m+1}\big(k_2^{h-(a+b)}r\big) + \frac{m}{r} J_m\big(k_2^{h-(a+b)}r\big)\right) \tag{77}$$

$$a_{10,10} = -\kappa^2\big(h-(a+b)\big)\mu \frac{1}{r} \, im \, J_m\big(k_3^{h-(a+b)}r\big) \tag{78}$$

$$b_{10} = 0 \tag{79}$$