Appendix A: The classical AK-MCS method

The Kriging method regards the g-function as an n-dimensional Gaussian random field, which reads:

$$g(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{x})^T \boldsymbol{\beta} + Z(\boldsymbol{x}) \tag{A1}$$

where $\boldsymbol{f}(\boldsymbol{x})^T \boldsymbol{\beta}$ refers to an linear regression model with a set of functional basis $\boldsymbol{f} = (f_1, \ldots, f_d)$, $Z(\boldsymbol{x})$ indicates a n-dimensional Gaussian random field with zero mean and variance σ_g^2. The covariance function of the random field for any two point $\boldsymbol{x}^{(i)}$ and $\boldsymbol{x}^{(j)}$ is given as:

$$C_{gg}\left(\boldsymbol{x}^{(i)}, \boldsymbol{x}^{(j)}\right) = \sigma_g^2 R\left(\boldsymbol{x}^{(i)}, \boldsymbol{x}^{(j)}\right) \tag{A2}$$

where $R\left(\boldsymbol{x}^{(i)}, \boldsymbol{x}^{(j)}\right)$ is the autocorrelation function, and the most commonly used one is the exponential one given by:

$$R\left(\boldsymbol{x}^{(i)}, \boldsymbol{x}^{(j)}\right) = \exp\left(-\sum_{k=1}^{n} \theta_k \left(x_k^{(i)} - x_k^{(j)}\right)^2\right) \tag{A3}$$

and θ_k is a parameter measuring the strength of the autocorrelation. Then given a set of N_0 training samples $\left(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(N_0)}\right)^T$ and the corresponding values $\boldsymbol{z} = \left(g\left(\boldsymbol{x}^{(1)}\right), \ldots, g\left(\boldsymbol{x}^{(N_0)}\right)\right)^T$ of the gfunction, the mean value $\mu_{\hat{g}}(\boldsymbol{x})$ and the variance $\sigma_{\hat{g}}^2(\boldsymbol{x})$ for a new point $\boldsymbol{x}$ conditional on these training samples are given as [24]:

$$\begin{cases} \mu_{\hat{g}}(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{x})^T \hat{\boldsymbol{\beta}} + \boldsymbol{r}(\boldsymbol{x})^T \mathbf{R}^{-1}\left(\boldsymbol{z} - \mathbf{F}\hat{\boldsymbol{\beta}}\right) \\ \sigma_{\hat{g}}^2(\boldsymbol{x}) = \sigma_g^2\left(1 - \boldsymbol{r}(\boldsymbol{x})^T \mathbf{R}^{-1} \boldsymbol{r}(\boldsymbol{x}) + \boldsymbol{u}(\boldsymbol{x})^T \left(\mathbf{F}^T \mathbf{R}^{-1} \mathbf{F}\right)^{-1} \boldsymbol{u}(\boldsymbol{x})\right) \end{cases} \tag{A4}$$

where $\mathbf{R}$ is the correlation matrix of these N_0 training samples, $\boldsymbol{r}(\boldsymbol{x})$ refers to the vector of correlation functions between $\boldsymbol{x}$ and the N_0 training samples, $\mathbf{F}$ indicates the regression matrix computed by $F_{ij} = f_j\left(\boldsymbol{x}^{(i)}\right)$ with $i = 1, \ldots, N$ and $j = 1, \ldots, d$, $\boldsymbol{u}(\boldsymbol{x}) = \mathbf{F}^T \mathbf{R}^{-1} \boldsymbol{r}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{x})$ and the least square estimates of the regression coefficients are computed by $\hat{\boldsymbol{\beta}} = \left(\mathbf{F}^T \mathbf{R}^{-1} \mathbf{F}\right)^{-1} \mathbf{F}^T \mathbf{R}^{-1} \boldsymbol{z}$.

Then $\mu_{\hat{g}}(\boldsymbol{x})$ refers to the Kriging surrogate model, and $\sigma_{\hat{g}}^2(\boldsymbol{x})$ indicates the mean square error (MSE) of the estimator $\mu_{\hat{g}}(\boldsymbol{x})$. It holds that $\mu_{\hat{g}}(\boldsymbol{x}^{(j)}) = g(\boldsymbol{x}^{(j)})$ and $\sigma_{\hat{g}}^2(\boldsymbol{x}^{(j)}) = 0$ for $j = 1,2,\ldots,N_0$

The AK-MCS procedure for estimating the failure probability is summarized as follows [24]:

Step 1: Generate a MC population **W** of N (N is large, i.e., 1e5) samples of the input vector. Randomly chose N_0 (N_0 is small, i.e., 12) samples from the population **W**. Estimate the corresponding $g$function values for these N_0 training samples, and attribute these N_0 samples to the training population $\mathbf{W}_t$.

Step 2: Train the Kriging model based on the training population $\mathbf{W}_t$ using the DACE toolbox [24].

Step 3: Generate the Kriging prediction $\mu_{\hat{g}}(\boldsymbol{x}^{(j)})$ and $\sigma_{\hat{g}}^2(\boldsymbol{x}^{(j)})$ for each of the remaining N-N_0 MC samples based on the well-trained Kriging model, and then estimate the value of the U function for each of these N-N_0 MC samples by $U_j = U(\boldsymbol{x}^{(j)}) = \left|\mu_{\hat{g}}(\boldsymbol{x}^{(j)})\right| / \sigma_{\hat{g}}^2(\boldsymbol{x}^{(j)})$. If $\min_{j=1}^{N} U_j < 2$, then add the sample with the minimum U value to the training population $\mathbf{W}_t$, let $N_0 = N_0+1$, go to step 2; otherwise, estimate the failure probability as well as the C.O.V. of the estimate based on the Kriging predictions of the N MC samples contained in **W**.

Step 4: If the C.O.V. of the estimate of failure probability is greater than 0.05, then update **W** with a new MC population, and go to step 3; otherwise, end the procedure.

Appendix B: IS Estimators of the GRS indices

Let x'_j indicate the random replication of x_j ($j = 1,2,\ldots,n$), $\boldsymbol{x}'_{\sim i} = (x'_1,\ldots,x'_{i-1}, x_i, x'_{i+1},\ldots,x'_n)^T$, $\boldsymbol{x}'_i = (x_1,\ldots,x_{i-1}, x'_i, x_{i+1},\ldots,x_n)^T$ and $\boldsymbol{x}' = (x'_1, x'_2,\ldots,x'_n)^T$. As $E[I_F(\boldsymbol{x}) \mid x_i] = E[I_F(\boldsymbol{x}'_{\sim i}) \mid x_i]$ and $E[I_F(\boldsymbol{x})] = E[I_F(\boldsymbol{x}')]$, the main partial variance V_i can be derived as:

$$V_i = \int_{R^n} E[I_F(\boldsymbol{x}) \mid x_i]\, E[I_F(\boldsymbol{x}'_{\sim i}) \mid x_i]\, p(x_i)\, dx_i - E[I_F(\boldsymbol{x})]\, E[I_F(\boldsymbol{x}')]$$

□□$_{R^{2n-1}}$ I_F □ □$\boldsymbol{x}$ $I_F \boldsymbol{x}$□' $_i$□ p x_k □ $_k$ □dx_k □ p x_k □ $_{k'}$ □d$x_{k'}$

k□1 k□ □1,k i

□□□□$_{R^n}$ I_F □ □$\boldsymbol{x}$ □$_n$ p x_k □ $_k$ □dx_k □□□□□$_{R^n}$ I_F □$\boldsymbol{x}'$ □□$_n$ p x_k □ $_{k'}$ □d$x_{k'}$ □□

□ k□1 □□ k□1 □

□□□ □□$_{R^{2n}}$ □□I_F □ □$\boldsymbol{x}$

□$_{n1}$ h xp x_k □ $_k$ □□□□□□I_F □$\boldsymbol{x}'$ □□$_{n1}$ h xp x_k □□ $_{'k'}$ □□□□□ $_{n1}$ □h x

h x_k □ $_k$ □ $_k$ □ $_{k'}$ □dx x_kd $_{k'}$ □□

□□$_{R^{2n}}$ □□□□I_F □ □$\boldsymbol{x}$ □$_{kn-1}$ h xp x_{kk} □□ $_{kk}$ □□□□□□□□□□I_F □$\boldsymbol{x}$□' $_i$ □ $_j$□ □□$_{1n,j\,i}$ h xp x_{jj} □□ $_{k'k'}$ □□□□□ $_{kn-1}$

□h x h x_k □ $_k$ □ $_k$ □ $_{k'}$ □dx x_kd $_{k'}$ □□

□

□□ $_k$□ $_k$ □ $_k$ □□□□ $_k$□ $_k$ $_k$ □ $_k$□ □

$_n$ p x_k □ $_k$ □□ $_n$ p x_k □ $_{k'}$ □ $_n$ p x_k □ $_{k'}$ □□ $_n$ □h x h

F □ □I $\boldsymbol{x}$ □

□□$_{R^{2n}}$ I_F □ □$\boldsymbol{x}$ □$_{k-1}$ h x_k □ $_k$ □ □□□I □$\boldsymbol{x}$□' $_i$ □ $_k$□ □1,k i h x_k □ $_{k'}$ □$_F$ □ ' □□$_{k-1}$ h x_k □ $_{k'}$ □□□□ $_{k-1}$ □ $_k$ □ $_k$ □ $_k$ □$x_{k'}$

□dx x_kd $_{k'}$ □□

□ $_n$ p x_k □ $_k$ □□ ' $_n$ p x_k □ $_{k'}$ □ '

$_n$ p x_k □ $_{k'}$ □□□□

□

□ E$_h$ □I_F □ □$\boldsymbol{x}$ □ □I

□□$_{k-1}$ h x_k □ $_k$ □ □□ $_F$ □$\boldsymbol{x}$□$_i$ □ $_k$□ □□$_{1,k\,i}$ h x_k □ $_{k'}$ □ □I_F □$\boldsymbol{x}$ □□$_{k-1}$ h x_k □ $_{k'}$ □□□□□□

(A5)

Then the sample matrices **B**, **A** and **C** introduced in subsection 3.4 can be regarded as the IS samples of the random vectors $\boldsymbol{x}$, $\boldsymbol{x}'$ and $\boldsymbol{x}_{\sim i}'$, and the IS estimator of V_i is given as:

V_i

$\hat{\ } \; 1 \; _{N_{IS}} \; I_F \; b_j \; _n \; hp_{kk} \; bb_{jkjk}$

$I_F \; c_j \; _{k} \; _{1n,k\,i} \, hp_{kk} \; cc_{jkjk} \; I_F \; a_j \; _{kn1} \, hp_{kk} \; aa_{jkjk}$ (A6)

$N_{IS\,j1} \quad k1$

Similarly, as E I_F^2 x =E I_F^2 $x_{i'}$ and E I_F $x\,x|\,_i$ E I_F $x\,x_{i'}$

$|\,_i$, the total partial variance V_{Ti} can be derived as:

V_{Ti} E I_{F2} E E $_2$ $I_F\,|\,x_{i}$

$\frac{1}{2}$ E I_{F2} x $\frac{1}{2}$ E I_{F2} $x_{i'}$ E E I_F $x\,x|\,_i$ E I_F

$x\,x_{i'}$ $|\,_i$

$\frac{1}{2}\;{}_n p\,x_k\;{}_k \quad \frac{1}{2}\;{}' p\,x_i\;{}_{i'}$

$_n\; p\,x_k\;{}_k$

$R_n\, I_F \quad x \quad {}_1 h \quad x_k\;{}_k\; h \quad x_k\;{}_k\,\mathrm{d}x_k \;\; 2\, R_n\, I_F\; x_i \quad h\,x_i$

${}_{i'}\, h\,x_i \;\; {}_i \mathrm{d}x_{i\,k} \;\; {}_{1,k\,i}\, h\,x_k\;{}_k\; h\,x_k\;{}_k\,\mathrm{d}x_k$

$2\;{}_k\; R_{n1}\, I_F\; x\; I_F\; x_{i'}\; h\,xp\,x_{ii}\;\; {}_{i'i'}\, h\,x_i\;\; {}_{i'}\;\; \mathrm{d}x_{i'}\;\; {}_{kn1}\, h\,xp\,x_{kk}$

${}_{kk}\quad h\,x_k\;{}_k\,\mathrm{d}x_k$

$\frac{1}{2}\;{}_n\; p\,x_k\;{}_k\;{}_2\; p\,x_i\;{}_i\;{}_2\;{}' p\,x_i\;{}_{i'}\;{}' p\,x_i\;{}_{i'}\, p\,x_i\;{}_i\;{}'\;{}'\;{}_n$

$R_{n1}\; h\,x \quad I_F \quad x\, h\,x \quad I_F\; x_i\;{}'\; 2I_F\; x\; I_F$

$x_i \quad {}' h\,x\; h\,x_i\;\; {}_i \quad \mathrm{d}x_i\; h\,x_k\;{}_k\,\mathrm{d}x_k$

${}_{k1,k\,ik}\;{}_k \quad {}_i\;{}_i \quad h\,x_i\;{}_i \quad h\,x_i\;{}_{ii}\;{}_i \quad {}_{k1}$

$\frac{1}{2}$ E$_h$ $_{k}$ $_{1n,k\,i}\, h$

$xp\,x_{kk}\;{}_{kk}\; I_F\; x\; h\,xp\,x_{ii}\;\; {}_{ii}\, I_F\; x_{i'}\; h\,xp\,x_{ii}\;\; {}_{i'i'}\; 2I_F\; x$

$I_F\; x_{i'}\; h\,xp\,x_{ii}\;\; {}_{'i'}\, h\,xp\,x_{ii}\;\; {}_{ii}$

i

(A7)

Then the sample matrices **A** and **C** can be regarded as the IS sample of the vectors x and

IS estimator of V_{Ti} is derived as:

$\boldsymbol{x}_i^{'}$, and the

$\hat{V}_{Ti}$ □ $\frac{1}{N}$ □$^{N_{IS}}$□□ □ □□$_1^n$ $h p_{kk}$ □□ $a a_{jkjk}$ □□□□□□□□□□ I_F □$\boldsymbol{a}_j$□ $h\ ap\ a_{ii}$ □□

$_{jiji}$ □□ □ I_F □$\boldsymbol{c}_j$□ $h\ cp\ c_{ii}$ □□ $_{jiji}$ □□ □$2I_F$ □$\boldsymbol{a}_j$ □I_F □$\boldsymbol{c}_j$□ $h\ ap\ a_{ii}$ □□ $_{jiji}$

□□ $h\ cp\ c_{ii}$ □□ $_{jiji}$ □□□□□□

2 $_{IS\ j□1}$ □□$_{k\ ,k\ i}$

(A8)

Appendix C: FORM estimators of the GRS indices

Assume that the random input vector follows independent Gaussian distribution with mean vector $\boldsymbol{\mu}$ □□□ □$_1$, $_2$,…,□$_n$ □^{T} and SD vector $\boldsymbol{\sigma}$ □□□□$_1$, $_2$,…,□$_n$ □^{T}. The first order Taylors series of the limit state function expanded at the MPP $\boldsymbol{x}^*$ □□x_1^*, x_2^*,…, x_n^* □^{T} is given as follows:

$$z□\ □\boldsymbol{x}\ □\ g□\boldsymbol{x}^*\ □□□\sum_{j□1}^{n} a_j\ x_j\ □\ x_j^*\ □\ □\ a_0\ □\sum_{j□1}^{n} a_j\ x_j \tag{A9}$$

where a_j indicates the partial derivatives of the g-function w.r.t. x_j at the MPP, and

a_0 □ g□$\boldsymbol{x}^*$□□$\sum_{j□1}^{n} a_j\ x_j^*$.

Then, based on the first line of Eq. (A5), the main partial variance V_i can be derived as:

$$V_i\ □\ □I_F\ □\ □\boldsymbol{x}\ I_F\ □\boldsymbol{x}_{□' i}\ □□□\prod_{j□1}^{n} p_j\ □x_j\ □\mathrm{d}x_j\ □□\ □\ □\prod_{j□1, j\neq i}^{n} p_j\ □x_j'\ □\mathrm{d}x_j'\ □□□P_f^2 \tag{A10}$$

$$□\ \Pr□z□\ □\boldsymbol{x}\ □\ 0\cap z□\boldsymbol{x}_{□' i}□□\ 0□□P_f^2$$

where z□$\boldsymbol{x}$□ and z□$\boldsymbol{x}_{□' i}$□ are two linear function of the vector $\boldsymbol{x}$ and $\boldsymbol{x}_{□' i}$, respectively, thus can be regarded as two correlated Gaussian random variables with covariance □$_{zm}^2$ □a_i^2□$_i^2$. z□$\boldsymbol{x}$□ and z□$\boldsymbol{x}_{□' i}$□ have the same mean value □$_z$ □ □a_0 □$\sum_{j□1}^{n} a_j$□$_j$ and the same SD □$_z$ □ $\sqrt{□\sum_{j□1}^{n} a_j^2 □_j^2}$. Then V_i can be estimated by:

$$\hat{V}_i\ □□_2\ □[0,0];□_m,□\ □_m\ □\ \hat{P}_f^2 \tag{A11}$$

where □$_2$ □[0,0];□$_m$,□$_m$□ indicates the bivariate joint CDF of Gaussian distribution with mean vector

$\boldsymbol{\mu}_m = [\mu_z, \mu_z]$ and covariance matrix $\boldsymbol{\Sigma}_m = \begin{bmatrix} \sigma_z^2 & \sigma_{zm}^2 \\ \sigma_{zm}^2 & \sigma_z^2 \end{bmatrix}$ calculated at the point [0,0].

Similarly, based on the first line of Eq.(A7), the total partial variance V_{Ti} can be approximated by:

$$V_{Ti} = P_f - \int \left[I_F(\boldsymbol{x}) I_F(\boldsymbol{x}_i') \prod_{j=1}^{n} p_j(x_j) dx_j \prod_{j=1, j \neq i}^{n} p_j(x_j') dx_j' \right] \tag{A12}$$
$$= P_f - \Pr\left\{ z(\boldsymbol{x}) \le 0 \cap z(\boldsymbol{x}_i') \le 0 \right\}$$

where $z(\boldsymbol{x})$ and $z(\boldsymbol{x}_i')$ are linear functions of $\boldsymbol{x}$ and $\boldsymbol{x}_i'$, thus are also two correlated Gaussian random variables with covariance $\sigma_{zt}^2 = \sum_{j=1, j \neq i}^{n} a_j^2 \sigma_j^2$. Then, based on Eq. (A12), the total partial variance V_{Ti} can be further derived as:

$$\hat{V}_{Ti} = \hat{P}_f - \Phi_2\left([0,0]; \boldsymbol{\mu}_t, \boldsymbol{\Sigma}_t\right) \tag{A13}$$

where $\boldsymbol{\mu}_t = \boldsymbol{\mu}_m$ and $\boldsymbol{\Sigma}_t = \begin{bmatrix} \sigma_z^2 & \sigma_{zt}^2 \\ \sigma_{zt}^2 & \sigma_z^2 \end{bmatrix}$.

◆