Technical Appendix (Not Intended for Publication, Possible Online Appendix)

A Comparative Static Analysis

We begin by logarithmically differentiating the first-order conditions from section 2, which are given by

$$\pi_s(s,T,q) = \left[\frac{S_s(s,q)}{(r-g)}\right] e^{gT} - C_s(s,q) = 0, \tag{A.1}$$

$$\pi_T(s,T,q) = -S(s,q)e^{gT} + B(s,q) = 0. \tag{A.2}$$

The advantage of logarithmic differentiation is that all terms are expressed in unitless quantities such as elasticities. Totally differentiate (A.1) and (A.2) with respect to s, T, and q, holding the parameters r and g constant, where to simplify notation we suppress the arguments of the functions:

$$\left[\frac{S_{ss}}{(r-g)}e^{gT} - C_{ss}\right] ds + \left[\frac{gS_s}{(r-g)}e^{gT}\right] dT = \left[-\frac{S_{sq}}{(r-g)}e^{gT} + C_{sq}\right] dq, \tag{A.3}$$

$$\left[\frac{gS_s}{(r-g)}]e^{gT}\right] ds + \left[-gSe^{-gT}\right] dT = \left[S_q e^{-gT} - B_q\right] dq, \tag{A.4}$$

and where use is made of the fact that $\pi_{T_s} = -S_s(s,q)e^{gT} + B_s(s,q) = gS_s(s,q)e^{gT}/(r-g) = \pi_{T_s}$. Next divide each term by the corresponding term in the equation that was totally differentiated. Using the relationships derived from the first-order conditions in (A.3) and (A.4), the two equations become

$$\left[\frac{S_{ss}}{S_s} - \frac{C_{ss}}{C_s}\right] ds + [g]\, dT = \left[-\frac{S_{sq}}{S_s} + \frac{C_{sq}}{C_s}\right] dq, \tag{A.5}$$

$$\left[\frac{g}{r-g}\frac{S_s}{S}\right] ds + [-g]\, dT = \left[\frac{S_q}{S} - \frac{B_q}{B}\right] dq, \tag{A.6}$$

which may be rewritten as

$$\left[\frac{S_{ss}s}{S_s} - \frac{C_{ss}s}{C_s}\right]\hat{s} + [g]\,dT = \left[-\frac{S_{sq}q}{S_s} + \frac{C_{sq}q}{C_s}\right]\hat{q}, \tag{A.7}$$

$$\left[\frac{g}{r-g}\frac{S_s s}{S}\right]\hat{s} + [-g]\,dT = \left[\frac{S_q q}{S} - \frac{B_q q}{B}\right]\hat{q}, \tag{A.8}$$

where $\hat{s} = ds/s$, etc. Now,

$$\frac{B_q q}{B} = \left[\frac{R_q q}{R}\right]\left[\frac{R}{(r(C+L)+R)}\right] + \left[\frac{r(C_q+L_q)q}{(r(C+L)}\right]\left[\frac{r(C+L)}{(r(C+L)+R)}\right].$$

Define

$$\theta \equiv R/(r(C+L)+R), \tag{A.9}$$

so that

$$\frac{B_q q}{B} = \left[\frac{R_q q}{R}\right]\theta + \left[\frac{r(C_q+L_q)q}{r(C+L)}\right](1-\theta). \tag{A.10}$$

θ is the share of that cost of postponing development one period that is agricultural rent. Then (A.8) above may be written as

$$\left[\frac{g}{r-g}\frac{S_s s}{S}\right]\hat{s} + [-g]\,dT = \left[\frac{S_q q}{S} - \left[\frac{R_q q}{R}\right]\theta - \left[\frac{(C_q+L_q)q}{(C+L)}\right](1-\theta)\right]\hat{q}. \tag{A.11}$$

Rewrite (A.7) and (A.11) using elasticity notation (with $E_{a:b}$ denoting the elasticity of a with respect to b) which yields

$$[E_{S_s:s} - E_{C_s:s}]\hat{s} + [g]dT = [E_{S_s:q} - E_{C_s:q}]\hat{q}, \tag{A.12}$$

$$\left[\frac{g}{r-g}E_{S:s}\right]\hat{s} + [-g]dT = \left[E_{S:q} - E_{R:q}\theta - E_{(C+L):q}(1-\theta)\right]\hat{q}. \tag{A.13}$$

Putting the pair of linear equations in matrix form we then have

$$\begin{bmatrix} U & V \\ X & Y \end{bmatrix}\begin{bmatrix} \hat{s} \\ dT \end{bmatrix} = \begin{bmatrix} W \\ Z \end{bmatrix}\hat{q}, \tag{A.14}$$

where

$$U = E_{S_s:s} - E_{C_s:s} < 0, \tag{A.15}$$

$$V = g > 0, \tag{A.16}$$

$$W = -E_{S_s:q} + E_{C_s:q} \gtreqless 0, \tag{A.17}$$

$$X = \frac{g}{r-g} E_{S:s} > 0, \tag{A.18}$$

$$Y = -g < 0, \tag{A.19}$$

$$Z = E_{S:q} - \theta E_{R:q} - (1-\theta) E_{(C+L):q} \gtreqless 0. \tag{A.20}$$

Finally, solving out the pair of simultaneous linear equations using Cramer's Rule gives

$$\frac{\hat{s}}{\hat{q}} = \frac{(WY - VZ)}{(UY - VX)}, \tag{A.21}$$

$$\frac{dT}{\hat{q}} = \frac{(UZ - WX)}{(UY - VX)}. \tag{A.22}$$

For Eqs. (A.15)-(A.20) we have indicated the signs of the various terms. The signs of W and Z depend on elasticities with respect to land quality, which depend on the nature of land quality, and are therefore *a priori* ambiguous $\left(\gtreqless 0\right)$. Since we are considering land that has not yet been developed at the present time, which we have normalized to zero, T is positive. Finally, we assume that, as functions of structural density, there are increasing marginal costs to construction, $E_{C_s:s} > 0$.

Define

$$\Delta = UY - VX \tag{A.23}$$

$$= g[-(E_{S_s:s} - E_{Cs:s}) - \frac{g}{r-g} E_{S:s}].$$

The third of the second-order conditions requires that Δ be positive.

An increase in land quality causes either or both of the first-order conditions, (A.1) and (A.2), to shift. The nature of land quality determines which of the first-order conditions is shifted and by how much. This is the subject of the subsections that follow. We consider only one type of land quality at a time.

A.1 An improvement in land quality corresponds to an increase in agricultural fertility

This is the simplest case to treat. Since quality is an ordinal concept, we may cardinalize it as we wish. Here the simplest cardinalization is to measure land quality as agricultural fertility, i.e. $R(q) = q$, in which case $E_{R:q} = 1$. Since only one type of land quality is considered at a time, $E_{S:q} = E_{C_s:q} = E_{(C+L):q} = 0$. We then obtain

$$\frac{\hat{s}}{\hat{q}} = \frac{\theta g}{\Delta} > 0, \tag{A.24}$$

$$\frac{dT}{\hat{q}} = -\frac{\theta[E_{S_s:s} - E_{C_s:s}]}{\Delta} > 0. \tag{A.25}$$

With respect to Fig. 1, an increase in agricultural fertility has no effect on FOC_s and causes FOC_T to shift to the right (since holding structural density fixed, profit-maximizing development time increases). Thus, more fertile land is developed later and at higher density, as argued earlier.

A.2 An improvement in land quality corresponds to a decrease in the fixed cost of servicing land per unit area

In this section we employ a different normalization of land quality in which land servicing cost is inversely proportional to quality, i.e. $L(q) = L/q$. Thus, $W = 0$ and $Z = -(1-\theta)E_{(C+L):q} = -(1-\theta)[d(C+L(q))/dq]q/(C+L) = (1-\theta)L/(C+L)$. We obtain:

$$\frac{\hat{s}}{\hat{q}} = -\frac{g(1-\theta)L}{(C+L)}\frac{1}{\Delta} < 0, \tag{A.26}$$

$$\frac{dT}{\hat{q}} = -\frac{(1-\theta)L(E_{S_s:s} - E_{C_s:s})}{(C+L)}\frac{1}{\Delta} < 0. \tag{A.27}$$

With respect to Fig. 1, a decrease in (fixed) land servicing costs has no effect on FOC_s and causes FOC_T to shift to the left (since holding structural density fixed, profit-maximizing development time decreases). Thus, a decrease in land servicing costs causes development to occur earlier and at lower density.

A.3 An improvement in land quality corresponds to a decrease in construction costs

Because of the type of land quality improvement being considered, $L_q = 0$, so that $E_{(C+L):q} = C_q q/(C+L) = E_{C:q}(C/(C+L))$. In addition $W = E_{C_s:q}$ and $Z = -(1-\theta)E_{(C+L):q}$. Then

$$\frac{\hat{s}}{\hat{q}} = g\frac{\left[-E_{C_s:q} + (1-\theta)E_{C:q}\frac{C}{(C+L)}\right]}{\Delta}, \tag{A.28}$$

$$\frac{dT}{\hat{q}} = \frac{\left[-(1-\theta)(E_{S_s:s} - E_{C_s:s})E_{C:q}\frac{C}{(C+L)} - (\frac{g}{r-g}E_{S:s})E_{C_s:q}\right]}{\Delta}. \tag{A.29}$$

We present two results. The first formalizes the argument presented in the main body of the paper about the effects of a proportional decline in construction costs at all densities. The second gives an example in which an increase in land quality that lowers marginal construction costs more than average construction costs results in construction being delayed.

Result 1: If there are no land improvement costs (so that $L = 0$), if agricultural rent is zero (so that $R = 0$), and if higher-quality land corresponds to an an equiproportional reduction in construction costs for all levels of density (so that $C(s,q) = C(s)f(q)$, with $q > 0$, $f > 0$ and $f' < 0$) then $\hat{s}/\hat{q} = 0$ and $dT/\hat{q} < 0$.

Proof. Under the stated conditions, $E_{C:q} = E_{C_s:q} = f'q/f$, so that (A.28) and (A.29) reduce to

$$\frac{\hat{s}}{\hat{q}} = \frac{g(-E_{C_s:q} + E_{C:q})}{\Delta} = 0,$$

$$\frac{dT}{\hat{q}} = \frac{-(E_{S_s:s} - E_{C_s:s})E_{C:q} - E_{C_s:q}(\frac{g}{r-g}E_{S:s})}{\Delta} = (f'q/f)\frac{-(E_{S_s:s} - E_{C_s:s}) - (\frac{g}{r-g}E_{S:s})}{\Delta}.$$

Recalling that $\Delta = g[-(E_{S_s:s} - E_{C_s:s}) - (\frac{g}{r-g}E_{S:s})]$, the expression for $dT/\hat{q}$ reduces to

$$\frac{dT}{\hat{q}} = \frac{(f'q/f)}{g} < 0.$$

Thus, when $\Delta > 0$, the improvement in quality causes profit-maximizing construction to be brought forward. □

The last equation follows directly from the first-order conditions. Under the stated conditions, the first-order conditions remain satisfied if $e^{gT}/f(q)$ is equal to a constant. Total differentiation gives $gdT - (f'/f)dq = 0$ so that $dT/\hat{q} = (f'q/f)/g$.

Result 2: If there are no land improvement costs (so that $L = 0$), if agricultural rent is zero (so that $R = 0$), if rent per unit floor area is independent of density (so that $S(s) = ps$), and if higher land quality results in a reduction in construction costs that is proportionally higher at higher densities, then it may be profitable to develop higher quality land later.

Proof. The proof entails the construction of a numerical example. We assume a construction cost function of the form $C(s,q) = s^b e^{as/q}$, where $a > 0$ and $b > 0$. This cost function has the property that an increase in land quality results in a greater proportional reduction in construction costs the higher is structural density (i.e., $\hat{C}/\hat{q} = -as/q^2$). The partial derivatives of this cost function are

$$C_s = C(b/s + a/q),$$

$$C_{ss} = C_s(b/s + a/q) - Cb/s^2 = C((b/s + a/q)^2 - b/s^2),$$

$$C_q = -asC/q^2,$$

$$C_{sq} = -aC/q^2 - asC_s/q^2 = -(aC/q^2)(1 + b + as/q).$$

We shall choose other parameters such at $s = 1$ and $T = 0$ at the optimum, and normalize such that $q = 1$ in the base situation. Then at the base optimum $C_s/C = b + a$, $C_{ss}/C_s = ((b + a)^2 - b)/(b + a)$, $C_q/C = -a$, and $C_{sq}/C_s = -a(1 + b + a)/(b + a)$.

Under the stated conditions, the first-order conditions are

$$s : pe^{gT} - (r - g)C_s(s, q) = pe^{gT} - (r - g)s^b e^{as/q}(b/s + a/q) = 0,$$

$$T : -pse^{gT} + rC(s, q) = -pse^{gT} + rs^b e^{as/q} = 0.$$

These conditions are satisfied at $q = s = 1$ and $T = 0$ if

$$p - (r - g)e^a(b + a) = 0, \quad -p + re^a = 0, \tag{A.30}$$

and in particular with the parameters $p = re^a$ and $(r - g)(b + a) = r$, which we assume to hold.

The example satisfies the second-order conditions that $\pi_{ss} < 0$ and $\pi_{TT} < 0$. To establish the Result, it remains to prove that $\Delta > 0$ and $dT/\hat{q} > 0$. Now, under the stated conditions, $E_{S:s} = 1$ and $E_{S_s:s} = 0$, so that $\Delta = g(E_{C_s:s} - g/(r - g))$. Since $E_{C_s:s} = [(b+a)^2 - b]/(b+a)$, from (A.30), $g/(r-g) = r/(r-g) - 1 = b+a-1$, it follows that $\Delta = g(E_{C_s:s} - g/(r - g)) = b + a - b/(b + a) - (b + a) + 1 = a/(a + b) > 0$. By a similar line of reasoning

$$\frac{dT}{\hat{q}} = \frac{[E_{C_s:s}E_{C:q} - E_{C_s:q}(\frac{g}{r-g})]}{g(E_{C_s:s} - \frac{g}{r-g})}$$

$$= \frac{\frac{a}{a+b}((1+b+a)\frac{g}{r-g} - ((b+a)^2 - b))}{g\left(\frac{a}{a+b}\right)}$$

$$= \frac{\left(\frac{g}{r-g} - a\right)}{g},$$

where use is made of the parameter restrictions in (A.30). It follows directly that if $a < g/(r-g)$, $dT/d\hat{q} > 0$. □

Consider an example in which $a = 1$ and $g/r = 2/3$ implying that $dT/\hat{q} = 1/g$. If $g = 0.02$, then a 10% improvement in land quality results in construction occurring five years later.

A.4 An improvement in land quality corresponds to an increase in amenities

Define the reference situation to occur when $S(s,q) = sp(q)$ so that the floor rent on all stories is the same, and an increase in amenities results in a uniform increase in the floor rent on all stories. In this reference situation, (A.21) and (A.22) reduce to

$$\frac{\hat{s}}{\hat{q}} = g\frac{(E_{S_s:q} - E_{S:q})}{\Delta} = 0, \tag{A.31}$$

$$\frac{dT}{\hat{q}} = \frac{[(E_{S_s:s} - E_{C_s:s})E_{S:q} + E_{S_s:q}(\frac{g}{r-g}E_{S:s})]}{\Delta} \tag{A.32}$$

$$= \frac{(p'q/p)[-E_{C_s:s} + \frac{g}{r-g}]}{\Delta} = -\frac{(p'q/p)}{g} < 0,$$

with

$$\Delta = g[-E_{S_s:s} + E_{C_s:s} - \frac{g}{r-g}E_{S:s}] = g[E_{C_s:s} - \frac{g}{r-g}].$$

Thus, with $\Delta > 0$ the increase in amenities results in earlier development at unchanged density. These results have a straightforward explanation. In the above specification, an increase in amenities changes the developer's problem only in that the future is brought forward; it is profit maximizing for the developer to do exactly as he would have done before the amenity improvement, but earlier. In particular, q and T are related that such that $p(q)e^{gT}$ is constant, so that $dT/\hat{q} = -(p'q/p)g$.

We now investigate whether, with the general specification of the rent function, $S(s, q)$, it is possible for an increase in amenities to result in a postponement of development. Intuitively for this to occur, the increase in amenities must increase rents disproportionately at higher density. We assume a particular rentals function, and derive restrictions on the construction cost function such that an increase in amenities causes development to be delayed. The assumed form of the rentals function is $S(s, q) = m(s/q + s^2q^2)$, with $m > 0$, and the parameters are such that, with $q = 1$, the profit maximum occurs with $s = 1$ and at $T = 0$. Rent is higher on higher stories (at the profit maximum with $s = 1$, $E_{S_s:s} = 2/3$) and is proportionally higher the higher the level of amenity ($dE_{S_s:s}/dq > 0$). With this rentals function,

$$E_{S:s} = (1/q + 2sq^2)/(1/q + sq^2),$$

$$E_{S_s:s} = 2sq^2/(1/q + 2sq^2),$$

$$E_{S:q} = (-s/q + 2s^2q^2)/(s/q + s^2q^2),$$

$$E_{S_s:q} = (-1/q^2 + 4q^2)/(1/q + 2sq^2).$$

Substituting these elasticities, evaluated at $q = 1$ and $s = 1$, gives the following expression for Δ and the numerator of $dT/\hat{q}$, which we define as N, from (A.32):

$$\Delta = -g\left(\left(\frac{2}{3} - E_{C_s:s}\right) + \frac{3}{2}\frac{g}{r-g}\right) = -g\left(\frac{2}{3} - E_{C_s:s} + \frac{3}{2}\frac{g}{r-g}\right), \tag{A.33}$$

$$N = \left(\frac{1}{2}\left(\frac{2}{3} - E_{C_s:s}\right) + \frac{3}{2}\frac{g}{r-g}\right) = \frac{1}{2}\left(\frac{2}{3} - E_{C_s:s} + 3\frac{g}{r-g}\right). \tag{A.34}$$

Both Δ and N must be positive for the second-order condition to be satisfied and for an increase in amenities to delay profit-maximizing development. Solving (A.33) and (A.34) for $\Delta > 0$ and $N > 0$ yields

$$\frac{2}{3} + \frac{3}{2}\frac{g}{r-g} < E_{C_s:s} < \frac{2}{3} + 3\frac{g}{r-g}. \tag{A.35}$$

Consider the case where $g/r = 1/2$, in which case $g/(r-g) = 1$ and $13/6 < E_{C_s:s} < 22/6$. Suppose $E_{C_s:s} = 3$, which falls in the appropriate range. This yields $dT/\hat{q} = 2/(5g)$. With a rental growth rate of 2%, a 10% increase in amenities yields a development delay of 2 years.