Supplementary Material to

Two-Way Partial AUC and Its Properties

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The supplementary material is organized as follows:

- In Section S.1, we provide key lemmas to the proof of main results.
- In Section S.2, we prove Theorem 4.1.
- In Section S.3, we prove Theorem 4.2.
- In Section S.4, we prove Theorem 4.3.
- In Section S.5, we prove Theorem 4.4.
- In Section S.6, we provide additional simulation results.

S.1 Key Lemmas

Lemma S.1.1. *Let m and n be sequences of integers such that $\frac{m}{m+n} \to \lambda$, $0 < \lambda < 1$, as $m, n \to \infty$; $F(t)$, $G(t)$ be continuous; $F^{-1}(1-q_0)$ be the unique solution of $F(-t) < 1-q_0 < F(t)$, $0 < q_0 < 1$. Then,*

$$\sqrt{m+n} \int_{F^{-1}(1-q_0)}^{F_m^{-1}(1-q_0)} \{F_m(t) - F(t)\} dG(t) = o_p(1), \quad m, n \to \infty. \tag{S.1}$$

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Proof. We can easily see that,

$$\left|\sqrt{m+n}\int_{F^{-1}(1-q_0)}^{F_m^{-1}(1-q_0)}\{F_m(t)-F(t)\}dG(t)\right| \le \left|\int_{F^{-1}(1-q_0)}^{F_m^{-1}(1-q_0)}\sqrt{m+n}\sup_t|F_m(t)-F(t)|dG(t)\right|$$
$$=\sup_t\sqrt{m+n}|F_m(t)-F(t)|\left|\int_{F^{-1}(1-q_0)}^{F_m^{-1}(1-q_0)}dG(t)\right| \quad \text{(S.2)}$$

Firstly,

$$\sup_t\sqrt{m+n}|F_m(t)-F(t)| = \sqrt{\frac{m+n}{m}}\sup_t\sqrt{m}|F_m(t)-F(t)|.$$

Because

$$\sup_t\sqrt{m}|F_m(t)-F(t)| = \mathbf{O}_p(1),$$

and

$$\sqrt{\frac{m+n}{m}} \to \sqrt{\frac{1}{\lambda}}, \quad m,n\to\infty.$$

Therefore

$$\sup_t\sqrt{m+n}|F_m(t)-F(t)| = \mathbf{O}_p(1), \quad m,n\to\infty. \quad \text{(S.3)}$$

Then we consider term $\left|\int_{F^{-1}(1-q_0)}^{F_m^{-1}(1-q_0)}dG(t)\right|$.

$$\begin{aligned}\int_{F^{-1}(1-q_0)}^{F_m^{-1}(1-q_0)}dG(t) =& G\{F_m^{-1}(1-q_0)\} - G\{F^{-1}(1-q_0)\}\\ =& G\{F^{-1}(1-q_0)\} - G_n\{F^{-1}(1-q_0)\} + \mathbf{O}(n^{-1})\\ =& o(1), \quad m\to\infty. \end{aligned} \quad \text{(S.4)}$$

Then apply (S.2) and (S.3) to (S.4), we have

$$\sqrt{m+n}\int_{F^{-1}(1-q_0)}^{F_m^{-1}(1-q_0)}\{F_m(t)-F(t)\}dG(t) = o_p(1), \quad m,n\to\infty.$$

□

Lemma S.1.2. *Let $F(t)$ and $G(t)$ be distribution functions with sample distribution functions $F_m(t)$*

and $G_n(t)$, *respectively;* $t \in \mathbf{R}$. *Let* $F(t)$ *be continuous. Let* $F^{-1}(1-q_0)$ *be the unique solution of* $F(t) \le 1-q_0 \le F(t)$, $0 < 1-q_0 < 1$. *Define* $f_m(t) = F_m\{F_m^{-1}(1-q_0)\}I\{t \le F_m^{-1}(1-q_0)\}$, *and* $f_0 = F\{F^{-1}(1-q_0)\}I\{t \le F^{-1}(1-q_0)\}$. *Then,*

$$\sqrt{m+n}(P_n f_m - P f_m - P_n f_0 + P f_0) = o_p(1), \quad m,n \to \infty.$$

Equivalently,

$$\sqrt{m+n}(P_n f_m - P f_0) = \sqrt{m+n}(P_n f_0 - P f_0) + \sqrt{m+n}(P f_n - P f_0) + o_p(1), \quad m,n \to \infty. \quad \text{(S.5)}$$

Proof. Let us consider the term $\mathbf{I}$ first. Since $G_n(x) = P_n((-\infty, x])$. In this case, the empirical process is indexed by a class $\mathcal{C} = \{(-\infty, x] : x \in \mathbf{R}\}$, with only one element in this class. It has been shown that $\mathcal{C}$ is a Donsker class, because $\sqrt{m+n}\{G_n(x) - G(x)\}$ converges weakly in $\mathcal{L}^\infty(\mathbf{R})$ to a Brownian bridge $B\{G(x)\}$. Thus, it is not difficult to conclude that $\mathcal{D} = \{F\{F^{-1}(1-q_0)\}I(t \le x) : x \in \mathbf{R}\}$ is also a Donsker class. Thus for $f_m(t) = F_m\{F_m^{-1}(1-q_0)\}I\{t \le F_m^{-1}(1-q_0)\}$, and $f_0 = F\{F^{-1}(1-q_0)\}I\{t \le F^{-1}(1-q_0)\}$, then they are in class $\mathcal{D}$.

Otherwise,

$$F_m^{-1}(1-q_0) \to F^{-1}(1-q_0) \quad wp1, \quad m \to \infty.$$

Then

$$\int [I\{t \le F_m^{-1}(1-q_0)\} - I\{t \le F^{-1}(1-q_0)\}] dP \xrightarrow{p} 0, \quad m \to \infty.$$

Also note that $\sup_t |H_m(t) - H(t)| \xrightarrow{p} 0, \quad m \to \infty$. Thus,

$$\int [f_m(t) - f(t)]^2 dP \xrightarrow{p} 0, \quad m \to \infty.$$

where H(t) is any distribution function.

Therefore by lemma 2.3 in (15), we can easily get

$$\sqrt{m+n}(P_n f_m - P f_m - P_n f_0 + P f_0) = o_p(1), \quad m,n \to \infty.$$

Equally,

$$\sqrt{m+n}(P_n f_m - Pf_0) = \sqrt{m+n}(P_n f_0 - Pf_0) + \sqrt{m+n}(Pf_n - Pf_0) + o_p(1), \quad m, n \to \infty.$$

□

Lemma S.1.3. *Let m and n be sequences of integers such that $\frac{m}{m+n} \to \lambda$, $0 < \lambda < 1$, as $m, n \to \infty$; $F(t)$, $G(t)$ be continuous; $F^{-1}(1-q_0)$ be the unique solution of $F(-t) < 1-q_0 < F(t)$, $0 < q_0 < 1$. Then,*

$$\sqrt{m+n}\int_{F^{-1}(1-q_0)}^{F_m^{-1}(1-q_0)} [F_m\{F_m^{-1}(1-q_0)\} - (1-q_0)]dG(t) = o_p(1), m, n \to \infty. \tag{S.6}$$

Proof. The proof methods are just exactly the same as Lemma S.1.1. □

Lemma S.1.4. *Let $G(t)$ be differentiable; and $G(t)$ be twice differentiable at $F^{-1}(1-q_0)$ and $G'\{F^{-1}(1-q_0)\} > 0$. Then,*

$$\begin{aligned}
&\sqrt{m+n}\left\{\int_{-\infty}^{F_m^{-1}(1-q_0)} F(t)dG(t) - \int_{-\infty}^{F^{-1}(1-q_0)} F(t)dG(t)\right\} \\
=&\sqrt{m+n}\frac{[1-q_0-F_m\{F^{-1}(1-q_0)\}]}{F'\{F^{-1}(1-q_0)\}}(1-q_0)G'\{F^{-1}(1-q_0)\} + o_p(1), \quad m \to \infty.
\end{aligned} \tag{S.7}$$

Proof. By Taylor Expansion, we have

$$\begin{aligned}
&\sqrt{m+n}\left\{\int_{-\infty}^{F_m^{-1}(1-q_0)} F(t)dG(t) - \int_{-\infty}^{F^{-1}(1-q_0)} F(t)dG(t)\right\} \\
=&\sqrt{m+n}\{F_m^{-1}(1-q_0) - F^{-1}(1-q_0)\}F\{F^{-1}(1-q_0)\}G'\{F^{-1}(1-q_0)\} + o_p(1), \quad m \to \infty.
\end{aligned}$$

Then by Theorem 2.13 in (15), we have

$$\sqrt{m}[F_m^{-1}(1-q_0) - F^{-1}(1-q_0)] = \sqrt{m}\frac{1-q_0-F_m\{F^{-1}(1-q_0)\}}{F'\{F^{-1}(1-q_0)\}} + o_p(1), \quad n \to \infty.$$

Thus

$$
\begin{aligned}
&\sqrt{m+n}\left\{\int_{-\infty}^{F_m^{-1}(1-q_0)} F(t)dG(t) - \int_{-\infty}^{F^{-1}(1-q_0)} F(t)dG(t)\right\} \\
=&\sqrt{m+n}\frac{1-q_0-F_m\{F^{-1}(1-q_0)\}}{F'\{F^{-1}(1-q_0)\}}(1-q_0)G^{'}\{F^{-1}(1-q_0)\}+o_p(1), \quad m\to\infty.
\end{aligned}
$$

□

Lemma S.1.5. *Let $G(t)$ be differentiable; and $G(t)$ be twice differentiable at $F^{-1}(1-q_0)$ and $G^{'}\{F^{-1}(1-q_0)\}>0$. Then,*

$$
\begin{aligned}
&\sqrt{m+n}\left\{\int_{-\infty}^{F_m^{-1}(1-q_0)} (1-q_0)dG(t) - \int_{-\infty}^{F^{-1}(1-q_0)} (1-q_0)dG(t)\right\} \\
=&(1-q_0)\sqrt{m+n}\frac{[1-q_0-F_m\{F^{-1}(1-q_0)\}]}{F^{'}\{F^{-1}(1-q_0)\}}G^{'}\{F^{-1}(1-q_0)\}+o_p(1).
\end{aligned} \tag{S.8}
$$

Proof. The proof methods are just exactly the same as Lemma S.1.4. □

S.2 Proof of Theorem 4.1

Proof of this part follows similar steps in (15). The main idea is continuing splitting term $\sqrt{m+n}(\hat{U}-U)$ until it is divided into two parts that only depend on m or n respectively. Here we only provide some major procedures, detailed deduction process can be referred to in the complementary material.

At first, we need to show:

$$
\begin{aligned}
&\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} I(Y_j \leq X_i) I\{X_i \leq S_{F,m}^{-1}(q_0)\} I\{Y_j \geq S_{G,n}^{-1}(p_0)\} \\
=&\frac{1}{n}\sum_{j=1}^{n}\frac{1}{m}\sum_{i=1}^{m} I(Y_j \leq X_i) I\{X_i \leq S_{F,m}^{-1}(q_0)\} I\{Y_j \geq S_{G,n}^{-1}(p_0)\} I\{Y_j \leq S_{F,m}^{-1}(q_0)\} \\
=&\frac{1}{n}\sum_{j=1}^{n}\int_{Y_j}^{S_{F,m}^{-1}(q_0)} dF_m(t) I\{S_{G,n}^{-1}(p_0) \leq Y_j \leq S_{F,m}^{-1}(q_0)\} \\
=&\frac{1}{n}\sum_{j=1}^{n}[F_m\{S_{F,m}^{-1}(q_0)\} - F_m(Y_j)] I\{S_{G,n}^{-1}(p_0) \leq Y_j \leq S_{F,m}^{-1}(q_0)\} \\
=&F_m\{S_{F,m}^{-1}(q_0)\} I\{S_{G,n}^{-1}(p_0) \leq Y_j \leq S_{F,m}^{-1}(q_0)\} - \frac{1}{n}\sum_{j=1}^{n} F_m(Y_j) I\{S_{G,n}^{-1}(p_0) \leq Y_j \leq S_{F,m}^{-1}(q_0)\} \\
=&\int_{G_n^{-1}(1-p_0)}^{F_m^{-1}(1-q_0)} F_m\{F_m^{-1}(1-q_0)\} dG_n(t) - \int_{G_n^{-1}(1-p_0)}^{F_m^{-1}(1-q_0)} F_m(t) dG_n(t) \\
=&\hat{U}(p_0, q_0). \qquad \text{(S.9)}
\end{aligned}
$$

On the other hand, we have:

$$
\begin{aligned}
U =& \int_{S_G\{S_F^{-1}(q_0)\}}^{p_0} S_F\{S_G^{-1}(u)\} du - [p_0 - S_G\{S_F^{-1}(q_0)\}] q_0 \\
=& \int_{S_F^{-1}(q_0)}^{S_G^{-1}(p_0)} S_F(t) dS_G(t) - [p_0 - S_G\{S_F^{-1}(q_0)\}] q_0 \\
=& q_0[S_G\{S_F^{-1}(q_0)\} - p_0] - \int_{S_G^{-1}(p_0)}^{S_F^{-1}(q_0)} S_F(t) dS_G(t) \\
=& q_0[1 - G\{F^{-1}(1-q_0)\} - p_0] + \int_{G^{-1}(1-p_0)}^{F^{-1}(1-q_0)} \{1 - F(t)\} dG(t) \\
=& (1-q_0)[G\{F^{-1}(1-q_0)\} - (1-p_0)] - \int_{G^{-1}(1-p_0)}^{F^{-1}(1-q_0)} F(t) dG(t). \qquad \text{(S.10)}
\end{aligned}
$$

It is obviously that both $\hat{U}$ and U can be composed of two parts. Thus it is natural to write the difference of $\hat{U}$ and U into two parts:

$$
\begin{aligned}
&\sqrt{m+n}(\hat{U}-U)\\
=&\sqrt{m+n}\left\{\int_{G_n^{-1}(1-p_0)}^{F_m^{-1}(1-q_0)} F_m\{F_m^{-1}(1-q_0)\}dG_n(t)-(1-q_0)[G\{F^{-1}(1-q_0)\}-(1-p_0)]\right\}\\
&-\sqrt{m+n}\left\{\int_{G_n^{-1}(1-p_0)}^{F_m^{-1}(1-q_0)} F_m(t)dG_n(t)-\int_{G^{-1}(1-p_0)}^{F^{-1}(1-q_0)} F(t)dG(t)\right\}\\
=&\sqrt{m+n}\underbrace{\left\{\int_{G_n^{-1}(1-p_0)}^{F_m^{-1}(1-q_0)} F_m\{F_m^{-1}(1-q_0)\}dG_n(t)-\int_{G^{-1}(1-p_0)}^{F^{-1}(1-q_0)} (1-q_0)dG_n(t)\right\}}_{\mathbf{I}}\\
&-\sqrt{m+n}\underbrace{\left\{\int_{G_n^{-1}(1-p_0)}^{F_m^{-1}(1-q_0)} F_m(t)dG_n(t)-\int_{G^{-1}(1-p_0)}^{F^{-1}(1-q_0)} F(t)dG(t)\right\}}_{\mathbf{II}}. \qquad \text{(S.11)}
\end{aligned}
$$

Firstly for term **II** in (S.11), we have

$$
\begin{aligned}
&\int_{G_n^{-1}(1-p_0)}^{F_m^{-1}(1-q_0)} F_m(t)dG_n(t)-\int_{G^{-1}(1-p_0)}^{F^{-1}(1-q_0)} F(t)dG(t)\\
=&\underbrace{\int_{-\infty}^{F_m^{-1}(1-q_0)} F_m(t)dG_n(t)-\int_{-\infty}^{F^{-1}(1-q_0)} F(t)dG(t)}_{\mathbf{I_1}}\\
&-\underbrace{\left\{\int_{-\infty}^{G_n^{-1}(1-p_0)} F_m(t)dG_n(t)-\int_{-\infty}^{G^{-1}(1-p_0)} F(t)dG(t)\right\}}_{\mathbf{II_1}}. \qquad \text{(S.12)}
\end{aligned}
$$

For term $\mathbf{I_1}$ in (S.12), according to equation (3.15) in (15),

$$
\begin{aligned}
&\sqrt{m+n}\left\{\int_{-\infty}^{G_n^{-1}(1-p_0)} F_m(t)dG_n(t)-\int_{-\infty}^{G^{-1}(1-p_0)} F(t)dG(t)\right\}\\
=&\sqrt{m+n}\int_{-\infty}^{G^{-1}(1-p_0)} F(t)d[G_n(t)-G(t)]+\sqrt{m+n}\int_{-\infty}^{G^{-1}(1-p_0)}[F_m(t)-F(t)]dG(t)\\
&+\sqrt{m+n}[1-p_0-G_n\{G^{-1}(1-p_0)\}]F\{G^{-1}(1-p_0)\}+o_p(1), m,n\to\infty. \qquad \text{(S.13)}
\end{aligned}
$$

For term $\mathbf{II_1}$, by Theorem 3.7 in (15), we get

$$\begin{aligned}
&\sqrt{m+n}\left\{\int_{-\infty}^{F_m^{-1}(1-q_0)} F_m(t)dG_n(t) - \int_{-\infty}^{F^{-1}(1-q_0)} F(t)dG(t)\right\} \\
=&\sqrt{m+n}\left\{\int_{-\infty}^{F^{-1}(1-q_0)} F(t)dG_n(t) - \int_{-\infty}^{F^{-1}(1-q_0)} F(t)dG(t)\right\} \\
&+ \underbrace{\sqrt{m+n}\left\{\int_{-\infty}^{F_m^{-1}(1-q_0)} F_m(t)dG(t) - \int_{-\infty}^{F^{-1}(1-q_0)} F(t)dG(t)\right\}}_{\mathbf{I_2}} + o_p(1). \qquad \text{(S.14)}
\end{aligned}$$

Continuing expanding the term $\mathbf{I_2}$ in (S.14), then we get

$$\begin{aligned}
&\sqrt{m+n}\left\{\int_{-\infty}^{F_m^{-1}(1-q_0)} F_m(t)dG(t) - \int_{-\infty}^{F^{-1}(1-q_0)} F(t)dG(t)\right\} \\
=&\sqrt{m+n}\int_{-\infty}^{F_m^{-1}(1-q_0)} \{F_m(t) - F(t)\}dG(t) \\
&+ \sqrt{m+n}\left\{\int_{-\infty}^{F_m^{-1}(1-q_0)} F(t)dG(t) - \int_{-\infty}^{F^{-1}(1-q_0)} F(t)dG(t)\right\} \\
=&\underbrace{\sqrt{m+n}\int_{F^{-1}(1-q_0)}^{F_m^{-1}(1-q_0)} \{F_m(t) - F(t)\}dG(t)}_{\mathbf{I_3}} + \sqrt{m+n}\int_{-\infty}^{F^{-1}(1-q_0)} \{F_m(t) - F(t)\}dG(t) \\
&+ \sqrt{m+n}\left\{\int_{-\infty}^{F_m^{-1}(1-q_0)} F(t)dG(t) - \int_{-\infty}^{F^{-1}(1-q_0)} F(t)dG(t)\right\}. \qquad \text{(S.15)}
\end{aligned}$$

Combining (S.12)-(S.15), and Lemma S.1.1, we have

$$
\begin{aligned}
&\int_{G_n^{-1}(1-p_0)}^{F_m^{-1}(1-q_0)} F_m(t)dG_n(t) - \int_{G^{-1}(1-p_0)}^{F^{-1}(1-q_0)} F(t)dG(t) \\
=&\sqrt{m+n}\int_{-\infty}^{G^{-1}(1-p_0)} F(t)d[G_n(t)-G(t)] + \sqrt{m+n}\int_{-\infty}^{G^{-1}(1-p_0)} [F_m(t)-F(t)]dG(t) \\
&+\sqrt{m+n}[1-p_0-G_n\{G^{-1}(1-p_0)\}]F\{G^{-1}(1-p_0)\} \\
&+\sqrt{m+n}\left\{\int_{-\infty}^{F^{-1}(1-q_0)} F(t)dG_n(t) - \int_{-\infty}^{F^{-1}(1-q_0)} F(t)dG(t)\right\} \\
&+\sqrt{m+n}\int_{-\infty}^{F^{-1}(1-q_0)} \{F_m(t)-F(t)\}dG(t) \\
&+\sqrt{m+n}\left\{\int_{-\infty}^{F_m^{-1}(1-q_0)} F(t)dG(t) - \int_{-\infty}^{F^{-1}(1-q_0)} F(t)dG(t)\right\} + o_p(1). \qquad \text{(S.16)}
\end{aligned}
$$

As for term **I** of (S.11), we have

$$
\begin{aligned}
&\int_{G_n^{-1}(1-p_0)}^{F_m^{-1}(1-q_0)} F_m\{F_m^{-1}(1-q_0)\}dG_n(t) - \int_{G^{-1}(1-p_0)}^{F^{-1}(1-q_0)} (1-q_0)dG(t) \\
=&\underbrace{\int_{-\infty}^{F_m^{-1}(1-q_0)} F_m\{F_m^{-1}(1-q_0)\}dG_n(t) - \int_{-\infty}^{F^{-1}(1-q_0)} (1-q_0)dG(t)}_{\mathbf{I_4}} \\
&-\underbrace{\left[\int_{-\infty}^{G_n^{-1}(1-p_0)} F_m\{F_m^{-1}(1-q_0)\}dG_n(t) - \int_{-\infty}^{G^{-1}(1-p_0)} (1-q_0)dG(t)\right]}_{\mathbf{II_4}}. \qquad \text{(S.17)}
\end{aligned}
$$

Then we apply Lemma S.1.2 to term $\mathbf{I_4}$ of (S.17), we have

$$
\begin{aligned}
&\sqrt{m+n}\left[\int_{-\infty}^{F_m^{-1}(1-q_0)} F_m\{F_m^{-1}(1-q_0)\}dG_n(t) - \int_{-\infty}^{F^{-1}(1-q_0)}(1-q_0)dG(t)\right] \\
=&\sqrt{m+n}\left\{\int_{-\infty}^{F^{-1}(1-q_0)}(1-q_0)dG_n(t) - \int_{-\infty}^{F^{-1}(1-q_0)}(1-q_0)dG(t)\right\} \\
&+\sqrt{m+n}\left[\int_{-\infty}^{F_m^{-1}(1-q_0)} F_m\{F_m^{-1}(1-q_0)\}dG(t) - \int_{-\infty}^{F^{-1}(1-q_0)}(1-q_0)dG(t)\right] + o_p(1) \\
=&\sqrt{m+n}\int_{-\infty}^{F^{-1}(1-q_0)}(1-q_0)d\{G_n(t)-G(t)\} \\
&+\underbrace{\sqrt{m+n}\left[\int_{-\infty}^{F_m^{-1}(1-q_0)} F_m\{F_m^{-1}(1-q_0)\}dG(t) - \int_{-\infty}^{F^{-1}(1-q_0)}(1-q_0)dG(t)\right]}_{\mathbf{I_5}} + o_p(1).
\end{aligned}
\tag{S.18}
$$

Similarly, for term $\mathbf{I_5}$ of (S.18), we have

$$
\begin{aligned}
&\sqrt{m+n}\left[\int_{-\infty}^{F_m^{-1}(1-q_0)} F_m\{F_m^{-1}(1-q_0)\}dG(t) - \int_{-\infty}^{F^{-1}(1-q_0)}(1-q_0)dG(t)\right] \\
=&\sqrt{m+n}\int_{-\infty}^{F_m^{-1}(1-q_0)}[F_m\{F_m^{-1}(1-q_0)\}-(1-q_0)]dG(t) \\
&+\sqrt{m+n}\left\{\int_{-\infty}^{F_m^{-1}(1-q_0)}(1-q_0)dG(t) - \int_{-\infty}^{F^{-1}(1-q_0)}(1-q_0)dG(t)\right\} \\
=&\underbrace{\sqrt{m+n}\int_{F^{-1}(1-q_0)}^{F_m^{-1}(1-q_0)}[F_m\{F_m^{-1}(1-q_0)\}-(1-q_0)]dG(t)}_{\mathbf{I_6}} \\
&+\sqrt{m+n}\int_{-\infty}^{F^{-1}(1-q_0)}[F_m\{F_m^{-1}(1-q_0)\}-(1-q_0)]dG(t) \\
&+\sqrt{m+n}\left\{\int_{-\infty}^{F_m^{-1}(1-q_0)}(1-q_0)dG(t) - \int_{-\infty}^{F^{-1}(1-q_0)}(1-q_0)dG(t)\right\}.
\end{aligned}
\tag{S.19}
$$

combining equations (S.18), (S.2) and Lemma S.1.3, we get

$$
\begin{aligned}
&\int_{-\infty}^{F_m^{-1}(1-q_0)} F_m\{F_m^{-1}(1-q_0)\}dG_n(t) - \int_{-\infty}^{F^{-1}(1-q_0)} (1-q_0)dG(t) \\
=&\sqrt{m+n}\int_{-\infty}^{F^{-1}(1-q_0)} (1-q_0)d\{G_n(t)-G(t)\} \\
&+\sqrt{m+n}\int_{-\infty}^{F^{-1}(1-q_0)} [F_m\{F_m^{-1}(1-q_0)\}-(1-q_0)]dG(t). \\
&+\sqrt{m+n}\left\{\int_{-\infty}^{F_m^{-1}(1-q_0)} (1-q_0)dG(t) - \int_{-\infty}^{F^{-1}(1-q_0)} (1-q_0)dG(t)\right\}
\end{aligned}
\tag{S.20}
$$

Then we apply the same process as (S.18), (S.2) and Lemma S.1.3 to term $\mathbf{II_4}$ of equation (S.17), we obtain

$$
\begin{aligned}
&\int_{-\infty}^{G_n^{-1}(1-p_0)} F_m\{F_m^{-1}(1-q_0)\}dG_n(t) - \int_{-\infty}^{G^{-1}(1-p_0)} (1-q_0)dG(t) \\
=&\sqrt{m+n}\int_{-\infty}^{G^{-1}(1-p_0)} (1-p_0)d[G_n(t)-G(t)] + \sqrt{m+n}\int_{-\infty}^{G^{-1}(1-p_0)} F_m\{F_m^{-1}(1-q_0)\}dG(t) \\
&+\sqrt{m+n}\left[(1-q_0)G\{G_n^{-1}(1-p_0)\} - 2(1-q_0)(1-p_0)\right].
\end{aligned}
\tag{S.21}
$$

Therefore, with the result of (S.20) and (S.21), (S.17) becomes

$$
\begin{aligned}
&\int_{G_n^{-1}(1-p_0)}^{F_m^{-1}(1-q_0)} F_m\{F_m^{-1}(1-q_0)\}dG_n(t) - \int_{G^{-1}(1-p_0)}^{F^{-1}(1-q_0)} (1-q_0)dG_n(t) \\
=&\sqrt{m+n}\int_{-\infty}^{F^{-1}(1-q_0)} (1-q_0)d\{G_n(t)-G(t)\} \\
&+\sqrt{m+n}\int_{-\infty}^{F^{-1}(1-q_0)} [F_m\{F_m^{-1}(1-q_0)\}-(1-q_0)]dG(t) \\
&+\sqrt{m+n}\left\{\int_{-\infty}^{F_m^{-1}(1-q_0)} (1-q_0)dG(t) - \int_{-\infty}^{F^{-1}(1-q_0)} (1-q_0)dG(t)\right\} \qquad \text{(S.22)} \\
&-(\sqrt{m+n}\int_{-\infty}^{G^{-1}(1-p_0)} (1-p_0)d[G_n(t)-G(t)] + \sqrt{m+n}\int_{-\infty}^{G^{-1}(1-p_0)} F_m\{F_m^{-1}(1-q_0)\}dG(t) \\
&+\sqrt{m+n}\left[(1-q_0)G\{G_n^{-1}(1-p_0)\} - 2(1-q_0)(1-p_0)\right]).
\end{aligned}
\tag{S.23}
$$

Thus, above all, (S.12) turns to

$$
\begin{aligned}
&\sqrt{m+n}(\hat{U}-U)\\
=&\sqrt{m+n}\int_{-\infty}^{F^{-1}(1-q_0)}(1-q_0)d\{G_n(t)-G(t)\}\\
&+\sqrt{m+n}\int_{-\infty}^{F^{-1}(1-q_0)}[F_m\{F_m^{-1}(1-q_0)\}-(1-q_0)]dG(t)\\
&+\underbrace{\sqrt{m+n}\left\{\int_{-\infty}^{F_m^{-1}(1-q_0)}(1-q_0)dG(t)-\int_{-\infty}^{F^{-1}(1-q_0)}(1-q_0)dG(t)\right\}}_{\mathbf{I_7}}\\
&-(\sqrt{m+n}\int_{-\infty}^{G^{-1}(1-p_0)}(1-q_0)d[G_n(t)-G(t)]\\
&+\sqrt{m+n}\int_{-\infty}^{G^{-1}(1-p_0)}[F_m\{F_m^{-1}(1-q_0)\}-(1-q_0)]dG(t)\\
&+(1-q_0)\underbrace{\sqrt{m+n}\left[G\{G_n^{-1}(1-p_0)\}-(1-p_0)\right]}_{\mathbf{II_7}})\\
&+\sqrt{m+n}\int_{-\infty}^{G^{-1}(1-p_0)}F(t)d[G_n(t)-G(t)]+\sqrt{m+n}\int_{-\infty}^{G^{-1}(1-p_0)}[F_m(t)-F(t)]dG(t)\\
&+\sqrt{m+n}[1-p_0-G_n\{G^{-1}(1-p_0)\}]F\{G^{-1}(1-p_0)\}\\
&-\sqrt{m+n}\left\{\int_{-\infty}^{F^{-1}(1-q_0)}F(t)dG_n(t)-\int_{-\infty}^{F^{-1}(1-q_0)}F(t)dG(t)\right\}\\
&-\sqrt{m+n}\int_{-\infty}^{F^{-1}(1-q_0)}\{F_m(t)-F(t)\}dG(t)\\
&\underbrace{-\sqrt{m+n}\left\{\int_{-\infty}^{F_m^{-1}(1-q_0)}F(t)dG(t)-\int_{-\infty}^{F^{-1}(1-q_0)}F(t)dG(t)\right\}}_{\mathbf{III_7}}.
\end{aligned}
\tag{S.24}
$$

On the other hand, with similar proof procedures to get Lemma S.1.2, we can reach the same conclusion,

$$
\sqrt{m+n}(P_mf_m-Pf_0)=\sqrt{m+n}(P_mf_0-Pf_0)+\sqrt{m+n}(Pf_m-Pf_0)+o_p(1),\quad m,n\to\infty. \tag{S.25}
$$

Then we apply (S.25) to term $\sqrt{m+n}[F_m\{F_m^{-1}(1-q_0)\} - (1-q_0)]$,

$$
\begin{aligned}
&\sqrt{m+n}[F_m\{F_m^{-1}(1-q_0)\} - (1-q_0)] \\
=&\sqrt{m+n}\left\{\int_{-\infty}^{F_m^{-1}(1-q_0)} dF_m(t) - \int_{-\infty}^{F^{-1}(1-q_0)} dF(t)\right\} \\
=&\sqrt{m+n}\left\{\int_{-\infty}^{F^{-1}(1-q_0)} dF_m(t) - \int_{-\infty}^{F^{-1}(1-q_0)} dF(t)\right\} \\
&+ \sqrt{m+n}\left\{\int_{-\infty}^{F^{-1}(1-q_0)} dF_m(t) - \int_{-\infty}^{F^{-1}(1-q_0)} dF(t)\right\} + o_p(1) \\
=&\sqrt{m+n}[F_m\{F^{-1}(1-q_0)\} - (1-q_0)] + \sqrt{m+n}[F\{F_m^{-1}(1-q_0)\} - (1-q_0)] + o_p(1). \qquad \text{(S.26)}
\end{aligned}
$$

Equivalently, it means

$$
\begin{aligned}
&\sqrt{m+n}F_m\{F_m^{-1}(1-q_0)\} \\
=&\sqrt{m+n}(1-q_0) + \sqrt{m+n}[F_m\{F^{-1}(1-q_0)\} - (1-q_0)] \\
&+ \underbrace{\sqrt{m+n}[F\{F_m^{-1}(1-q_0)\} - (1-q_0)]}_{\mathbf{I_8}} + o_p(1). \qquad \text{(S.27)}
\end{aligned}
$$

Moreover, we can apply the proof methods of Lemma S.1.4 and Lemma S.1.5 to term $\mathbf{II_7}$ of (S.24) and term $\mathbf{I_8}$ of (S.27), we have

$$
\begin{aligned}
&\sqrt{m+n}[F\{F_m^{-1}(1-q_0)\} - (1-q_0)] \\
=&\sqrt{m+n}\left\{\int_{-\infty}^{F_m^{-1}(1-q_0)} dF(t) - \int_{-\infty}^{F^{-1}(1-q_0)} dF(t)\right\} \\
=&\sqrt{m+n}[1-q_0 - F_m\{F^{-1}(1-q_0)\}] + o_p(1), \qquad \text{(S.28)}
\end{aligned}
$$

and

$$
\begin{aligned}
&\sqrt{m+n}[G\{G_n^{-1}(1-p_0)\}-(1-p_0)] \\
=&\sqrt{m+n}\left\{\int_{-\infty}^{G_n^{-1}(1-p_0)} dG(t)-\int_{-\infty}^{G^{-1}(1-p_0)} dG(t)\right\} \\
=&\sqrt{m+n}[1-p_0-G_n\{G^{-1}(1-p_0)\}]+o_p(1).
\end{aligned} \tag{S.29}
$$

Therefore, with (S.24)-(S.29), (S.24) becomes

$$
\begin{aligned}
&\sqrt{m+n}(\hat{U}-U) \\
=&\int_{F^{-1}(1-q_0)}^{G^{-1}(1-p_0)} F_m(t)dG(t)-\int_{F^{-1}(1-q_0)}^{G^{-1}(1-p_0)} F(t)dG(t) \\
&+(1-q_0)G_n\{F^{-1}(1-q_0)\}+\int_{-\infty}^{G^{-1}(1-p_0)} F(t)dG_n(t)-G_n\{G^{-1}(1-p_0)\}]F\{G^{-1}(1-p_0)\} \\
&-\int_{-\infty}^{F^{-1}(1-q_0)} F(t)dG_n(t) \\
&-[(1-q_0)G\{F^{-1}(1-q_0)\}+\int_{-\infty}^{G^{-1}(1-p_0)} F(t)dG(t)-(1-p_0)F\{G^{-1}(1-p_0)\} \\
&-\int_{-\infty}^{F^{-1}(1-q_0)} F(t)dG(t)]+o_p(1).
\end{aligned} \tag{S.30}
$$

Next, through integration by parts, we have

$$
\begin{aligned}
&\int_{-\infty}^{G^{-1}(1-p_0)} F(t)dG_n(t)-G_n\{G^{-1}(1-p_0)\}]F\{G^{-1}(1-p_0)\} \\
=&\int_{-\infty}^{G^{-1}(1-p_0)} G_n(t)dF(t),
\end{aligned} \tag{S.31}
$$

$$
\begin{aligned}
&\int_{-\infty}^{G^{-1}(1-p_0)} F(t)dG(t)-(1-p_0)F\{G^{-1}(1-p_0)\} \\
=&\int_{-\infty}^{G^{-1}(1-p_0)} G(t)dF(t),
\end{aligned} \tag{S.32}
$$

$$
\begin{aligned}
&(1-q_0)G_n\{F^{-1}(1-q_0)\} - \int_{-\infty}^{F^{-1}(1-q_0)} F(t)dG_n(t) \\
&= - \int_{-\infty}^{F^{-1}(1-q_0)} G_n(t)dF(t),
\end{aligned} \tag{S.33}
$$

and

$$
\begin{aligned}
&(1-q_0)G\{F^{-1}(1-q_0)\} - \int_{-\infty}^{F^{-1}(1-q_0)} F(t)dG(t) \\
&= - \int_{-\infty}^{F^{-1}(1-q_0)} G(t)dF(t).
\end{aligned} \tag{S.34}
$$

Then combining (S.30)-(S.34), we get

$$
\begin{aligned}
&\sqrt{m+n}(\hat{U} - U) \\
&= \int_{F^{-1}(1-q_0)}^{G^{-1}(1-p_0)} F_m(t)dG(t) - \int_{F^{-1}(1-q_0)}^{G^{-1}(1-p_0)} F(t)dG(t) \\
&\quad + \int_{F^{-1}(1-q_0)}^{G^{-1}(1-p_0)} G_n(t)dF(t) - \int_{F^{-1}(1-q_0)}^{G^{-1}(1-p_0)} G(t)dF(t) + o_p(1).
\end{aligned} \tag{S.35}
$$

For now we can write (S.35) into $\sqrt{m+n}(T_m - \mu_1 + T_n - \mu_2) + o_p(1)$, where

$$
T_m = \int_{F^{-1}(1-q_0)}^{G^{-1}(1-p_0)} F_m(t)dG(t),
$$

$$
\mu_1 = \int_{F^{-1}(1-q_0)}^{G^{-1}(1-p_0)} F(t)dG(t),
$$

$$
T_n = \int_{F^{-1}(1-q_0)}^{G^{-1}(1-p_0)} G_n(t)dF(t),
$$

and

$$
\mu_2 = \int_{F^{-1}(1-q_0)}^{G^{-1}(1-p_0)} G(t)dF(t).
$$

Consider T_m first, by rewriting, we have,

$$\begin{aligned}
T_m &= \int_{F^{-1}(1-q_0)}^{G^{-1}(1-p_0)} F_m(t)dG(t) \\
&= \int_{F^{-1}(1-q_0)}^{G^{-1}(1-p_0)} \frac{1}{m}\sum_{i=1}^{m} I(X_i < t)dG(t) \\
&= \frac{1}{m}\sum_{i=1}^{m}\int_{F^{-1}(1-q_0)}^{G^{-1}(1-p_0)} I(X_i < t)dG(t). \qquad \text{(S.36)}
\end{aligned}$$

Note that T_m is in fact a sum of i.i.d. random variable, thus

$$\begin{aligned}
E(T_m) =& E\left\{\int_{F^{-1}(1-q_0)}^{G^{-1}(1-p_0)} I(X < t)dG(t)\right\} \\
=& \int_{-\infty}^{\infty}\left\{\int_{F^{-1}(1-q_0)}^{G^{-1}(1-p_0)} I(X < t)dG(t)\right\} dF(X) \\
=& \int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} I\{F^{-1}(1-q_0) \le t \le G^{-1}(1-p_0)\}I(X < t)dG(t)\right\} dF(X) \\
=& \int_{-\infty}^{\infty} I\{F^{-1}(1-q_0) \le t \le G^{-1}(1-p_0)\}\left\{\int_{-\infty}^{\infty} I(X < t)dF(X)\right\} dG(t) \\
=& \int_{F^{-1}(1-q_0)}^{G^{-1}(1-p_0)} F(t)dG(t) \\
=& \mu_1. \qquad \text{(S.37)}
\end{aligned}$$

Denote σ_3^2 to be the variance of T_m, then

$$
\begin{aligned}
\sigma_3^2 =& Var\left\{\int_{F^{-1}(1-q_0)}^{G^{-1}(1-p_0)} I(X<t)dG(t)\right\} \\
=& E\left\{\int_{G^{-1}(1-p_0)}^{F^{-1}(1-q_0)} I(X<t)dG(t)\right\}^2 - \left[E\left\{\int_{G^{-1}(1-p_0)}^{F^{-1}(1-q_0)} I(X<t)dG(t)\right\}\right]^2 \\
=& \int_{-\infty}^{\infty}\left\{\int_{G^{-1}(1-p_0)}^{F^{-1}(1-q_0)} I(X<t)dG(t)\right\}^2 dF(X) - \left\{\int_{G^{-1}(1-p_0)}^{F^{-1}(1-q_0)} F(t)dG(t)\right\}^2 \\
=& \int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} I\{G^{-1}(1-p_0)\le t\le F^{-1}(1-q_0)\}I(X<t)dG(t)\right\}^2 dF(X) \\
& - \left\{\int_{G^{-1}(1-p_0)}^{F^{-1}(1-q_0)} F(t)dG(t)\right\}^2 \\
=& \int_{-\infty}^{\infty}[\int_{-\infty}^{\infty} I\{X\le G^{-1}(1-p_0)\}I\{G^{-1}(1-p_0)\le t\le F^{-1}(1-q_0)\}dG(t)+ \\
& + \int_{-\infty}^{\infty} I\{F^{-1}(1-q_0)>X>G^{-1}(1-p_0)\}I\{X\le t\le F^{-1}(1-q_0)\}dG(t)]^2 dF(X) \\
& - \left\{\int_{G^{-1}(1-p_0)}^{F^{-1}(1-q_0)} F(t)dG(t)\right\}^2 \\
=& \int_{-\infty}^{\infty}(I\{X\le G^{-1}(1-p_0)\}[G\{F^{-1}(1-q_0)\}-(1-p_0)]+ \\
& + I\{F^{-1}(1-q_0)>X>G^{-1}(1-p_0)\}\{G\{F^{-1}(1-q_0)\}-G(X)\})^2 dF(X) \\
& - \left\{\int_{G^{-1}(1-p_0)}^{F^{-1}(1-q_0)} F(t)dG(t)\right\}^2 \\
=& \int_{-\infty}^{\infty}(I\{X\le G^{-1}(1-p_0)\}[G\{F^{-1}(1-q_0)-(1-p_0)\}]^2+ \\
& + I\{F^{-1}(1-q_0)>X>G^{-1}(1-p_0)\}[G\{F^{-1}(1-q_0)\}-G(X)]^2)dF(X) \\
& - \left\{\int_{G^{-1}(1-p_0)}^{F^{-1}(1-q_0)} F(t)dG(t)\right\}^2 \\
=& \int_{-\infty}^{\infty}(I\{t\le G^{-1}(1-p_0)\}[G\{F^{-1}(1-q_0)\}-(1-p_0)]^2+ \\
& + I\{F^{-1}(1-q_0)>t>G^{-1}(1-p_0)\}[G\{F^{-1}(1-q_0)\}-G(t)]^2)dF(t) \\
& - \left\{\int_{G^{-1}(1-p_0)}^{F^{-1}(1-q_0)} F(t)dG(t)\right\}^2 \\
=& F\{G^{-1}(1-p_0)\}[G\{F^{-1}(1-q_0)\}-(1-p_0)]^2+ \\
& + \int_{G^{-1}(1-p_0)}^{F^{-1}(1-q_0)}[G\{F^{-1}(1-q_0)\}-G(t)]^2 dF(t) \\
& - \left\{\int_{G^{-1}(1-p_0)}^{F^{-1}(1-q_0)} F(t)dG(t)\right\}^2 .
\end{aligned}
$$

Then by Lindeberg-Levy Central Limit Theorem, we have

$$\sqrt{m}(T_m - \mu_1) \xrightarrow{d} N(0, \sigma_3^2), \quad m, n \to \infty.$$

Let m, n be sequences of integers such that $\frac{m}{m+n} \to \lambda$, further note that

$$\sqrt{\frac{n}{m}} = \sqrt{\frac{m+n}{m} - 1} \to \sqrt{\frac{1}{\lambda} - 1}, \quad m, n \to \infty.$$

Thus, by Slutsky's Theorem, we have,

$$\sqrt{n}(T_m - \mu_1) \xrightarrow{d} N\left\{0, \left(\frac{1}{\lambda} - 1\right)\sigma_3^2\right\}, \quad m, n \to \infty. \tag{S.38}$$

Then consider T_n,

$$\begin{aligned} T_n &= \int_{F^{-1}(1-q_0)}^{G^{-1}(1-p_0)} G_n(t) dF(t) \\ &= \int_{F^{-1}(1-q_0)}^{G^{-1}(1-p_0)} \frac{1}{n} \sum_{j=1}^{n} I(Y_j \le t) dF(t) \\ &= \frac{1}{n} \sum_{j=1}^{n} \int_{F^{-1}(1-q_0)}^{G^{-1}(1-p_0)} I(Y_j \le t) dF(t). \end{aligned} \tag{S.39}$$

Since T_n is also a sum of i.i.d. random variables, then

$$\begin{aligned} E(T_n) &= E\left\{\int_{F^{-1}(1-q_0)}^{G^{-1}(1-p_0)} I(Y \le t) dF(t)\right\} \\ &= \int_{-\infty}^{\infty} \left\{\int_{F^{-1}(1-q_0)}^{G^{-1}(1-p_0)} I(Y \le t) dF(t)\right\} dG(Y) \\ &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} I\{F^{-1}(1-q_0) \le t \le G^{-1}(1-p_0)\} I(Y \le t) dF(t)\right] dG(Y) \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} I\{Y \le t\} dG(Y) I\{F^{-1}(1-q_0) \le t \le G^{-1}(1-p_0)\} dF(t) \\ &= \int_{F^{-1}(1-q_0)}^{G^{-1}(1-p_0)} G(t) dF(t). \end{aligned} \tag{S.40}$$

Denote σ_4^2 to be the variance of T_n, then

$$
\begin{aligned}
\sigma_4^2 =& Var\left\{\int_{F^{-1}(1-q_0)}^{G^{-1}(1-p_0)} I(Y<t)dF(t)\right\}\\
=& E\left\{\int_{G^{-1}(1-p_0)}^{F^{-1}(1-q_0)} I(Y<t)dF(t)\right\}^2 - \left[E\left\{\int_{G^{-1}(1-p_0)}^{F^{-1}(1-q_0)} I(Y<t)dF(t)\right\}\right]^2\\
=& \int_{-\infty}^{\infty}\left\{\int_{G^{-1}(1-p_0)}^{F^{-1}(1-q_0)} I(Y<t)dF(t)\right\}^2 dG(Y) - \left\{\int_{G^{-1}(1-p_0)}^{F^{-1}(1-q_0)} G(t)dF(t)\right\}^2\\
=& \int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} I\{G^{-1}(1-p_0)\le t\le F^{-1}(1-q_0)\}I(Y<t)dF(t)\right\}^2 dG(Y)\\
&- \left\{\int_{G^{-1}(1-p_0)}^{F^{-1}(1-q_0)} G(t)dF(t)\right\}^2\\
=& \int_{-\infty}^{\infty}\{\int_{-\infty}^{\infty} I\{G^{-1}(1-p_0)\le t\le F^{-1}(1-q_0)\}I\{Y\le G^{-1}(1-p_0)\}dF(t)\\
&+ \int_{-\infty}^{\infty} I\{Y\le t\le F^{-1}(1-q_0)\}I\{G^{-1}(1-p_0)<Y<F^{-1}(1-q_0)\}dF(t)\}^2 dG(Y)\\
&- \left\{\int_{G^{-1}(1-p_0)}^{F^{-1}(1-q_0)} G(t)dF(t)\right\}^2\\
=& \int_{-\infty}^{\infty}\{[1-q_0-F\{G^{-1}(1-p_0)\}]I\{Y\le G^{-1}(1-p_0)\}\\
&+ \{1-q_0-F(Y)\}I\{G^{-1}(1-p_0)\le Y\le F^{-1}(1-q_0)\}\}^2 dG(Y)\\
&- \left\{\int_{G^{-1}(1-p_0)}^{F^{-1}(1-q_0)} G(t)dF(t)\right\}^2\\
=& \int_{-\infty}^{\infty}[1-q_0-F\{G^{-1}(1-p_0)\}]^2 I\{Y\le G^{-1}(1-p_0)\}\\
&+ \{1-q_0-F(Y)\}^2 I\{G^{-1}(1-p_0)\le Y\le F^{-1}(1-q_0)\})dG(Y)\\
&- \left\{\int_{G^{-1}(1-p_0)}^{F^{-1}(1-q_0)} G(t)dF(t)\right\}^2\\
=& \int_{-\infty}^{\infty}[1-q_0-F\{G^{-1}(1-p_0)\}]^2 I\{t\le G^{-1}(1-p_0)\}\\
&+ \{1-q_0-F(t)\}^2 I\{G^{-1}(1-p_0)\le t\le F^{-1}(1-q_0)\})dG(t)\\
&- \left\{\int_{G^{-1}(1-p_0)}^{F^{-1}(1-q_0)} G(t)dF(t)\right\}^2\\
=& [1-q_0-F\{G^{-1}(1-p_0)\}]^2(1-p_0)\\
&+ \int_{G^{-1}(1-p_0)}^{F^{-1}(1-q_0)}\{1-q_0-F(t)\}^2 dG(t)\\
&- \left\{\int_{G^{-1}(1-p_0)}^{F^{-1}(1-q_0)} G(t)dF(t)\right\}^2.
\end{aligned}
$$

Thus by Lindeberg-Levy Central Limit Theorem, we have

$$\sqrt{n}(T_n - \mu_2) \xrightarrow{d} N(0, \sigma_4^2), \quad m, n \to \infty. \tag{S.41}$$

Because T_m and T_n are independent, and $\hat{U} - U = T_m - \mu_1 + T_n - \mu_2$, thus

$$\sqrt{n}(\hat{U} - U) \xrightarrow{d} N\left\{0, \left(\frac{1}{\lambda} - 1\right)\sigma_3^2 + \sigma_4^2\right\}, \quad m, n \to \infty. \tag{S.42}$$

Since

$$\sqrt{\frac{m+n}{n}} = \frac{1}{\sqrt{\frac{n}{m+n}}} \to \sqrt{\frac{1}{1-\lambda}}, 0 < \lambda < 1, m, n \to \infty.$$

Then by Slutsky's Theorem,

$$\sqrt{m+n}(\hat{U} - U) \xrightarrow{d} N\left\{0, \frac{\sigma_3^2}{\lambda} + \frac{\sigma_4^2}{1-\lambda}\right\}, \quad m, n \to \infty, \tag{S.43}$$

where

$$\begin{aligned}
\sigma_3^2 =& F\{G^{-1}(1-p_0)\}[G\{F^{-1}(1-q_0)\} - (1-p_0)]^2 + \\
&+ \int_{G^{-1}(1-p_0)}^{F^{-1}(1-q_0)} [G\{F^{-1}(1-q_0)\} - G(t)]^2 dF(t) \\
&- \left\{\int_{G^{-1}(1-p_0)}^{F^{-1}(1-q_0)} F(t) dG(t)\right\}^2,
\end{aligned}$$

and

$$\begin{aligned}
\sigma_4^2 =& [1 - q_0 - F\{G^{-1}(1-p_0)\}]^2(1-p_0) \\
&+ \int_{G^{-1}(1-p_0)}^{F^{-1}(1-q_0)} \{1 - q_0 - F(t)\}^2 dG(t) \\
&- \left\{\int_{G^{-1}(1-p_0)}^{F^{-1}(1-q_0)} G(t) dF(t)\right\}^2.
\end{aligned}$$

Then we finished the proof Theorem 4.1.

S.3 Proof of Theorem 4.2

Firstly, for all those ROC area indexes $\hat{\theta}^*$ we referred to in this paper, we know $\sqrt{n+m}(\hat{\theta}^* - \theta)$ converges to a normal random variable in distribution. Secondly, according to equation (6.7) in Page 47 of (14), we have

$$v_{boot}^2 \rightarrow \frac{\sigma_3^2}{\lambda} + \frac{\sigma_4^2}{1-\lambda}, \quad B \rightarrow \infty.$$

Therefore we can then prove Theorem 4.2 directly by using Slutsky's Theorem.

S.4 Proof of Theorem 4.3

Our proof strategy is to apply Theorem 2 in (16), which establishes sufficient conditions for uniqueness and consistency for solution to likelihood estimation. To achieve this, we need to prove the four conditions in (16) are satisfied. We briefly introduce these conditions in our notation as follows,

(F1) $\partial S_{m,n}(\boldsymbol{\beta})/\partial \boldsymbol{\beta}$ exists and is continuous in $N_\delta(\boldsymbol{\beta}_0)$.

(F2) $\partial S_{m,n}(\boldsymbol{\beta})/\partial \boldsymbol{\beta} \rightarrow_p \mathbb{E}\{\partial S_{i,j}(\boldsymbol{\beta})/\partial \boldsymbol{\beta}\}$ uniformly in $N_\delta(\boldsymbol{\beta}_0)$, as $m, n \rightarrow \infty$.

(F3) With probability tends to one, as $m, n \rightarrow \infty$, $\partial S_{m,n}(\boldsymbol{\beta}_0)/\partial \boldsymbol{\beta}$ is negative definite.

(F4) $\mathbb{E} S_{m,n}(\boldsymbol{\beta}_0) = 0$.

We first prove Condition F3 is satisfied. From triangle inequality, we have, for any $\epsilon > 0$,

$$\begin{aligned}
&P\big\{|\partial S_{m,n}(\boldsymbol{\beta})/\partial \boldsymbol{\beta} - \mathbb{E}\{\partial S_{i,j}(\boldsymbol{\beta})/\partial \boldsymbol{\beta}\}| > \epsilon\big\} \\
\leq &P\big\{|\partial S_{m,n}(\boldsymbol{\beta})/\partial \boldsymbol{\beta} - \frac{1}{m}\sum_{i=1}^m \mathbb{E}\{\partial S_{i,j}(\boldsymbol{\beta})/\partial \boldsymbol{\beta}|X_i\}| > \epsilon/2\big\} \\
&+ P\big\{|\frac{1}{m}\sum_{i=1}^m \mathbb{E}\{\partial S_{i,j}(\boldsymbol{\beta})/\partial \boldsymbol{\beta}|X_i\} - \mathbb{E}\{\partial S_{i,j}(\boldsymbol{\beta})/\partial \boldsymbol{\beta}\}| > \epsilon/2\big\} \qquad \text{(S.44)}
\end{aligned}$$

where $\mathbb{E}\{\partial S_{i,j}(\boldsymbol{\beta})/\partial\boldsymbol{\beta}|X_i\}$ is independent across $i \in \{1,\dots,m\}$ and random in terms of X_i. For the first term in (S.44), we get

$$
\begin{aligned}
&\mathbb{E}\Big|\partial S_{m,n}(\boldsymbol{\beta})/\partial\boldsymbol{\beta} - \frac{1}{m}\sum_{i=1}^{m}\mathbb{E}\{\partial S_{i,j}(\boldsymbol{\beta})/\partial\boldsymbol{\beta}|X_i\}\Big| \\
=&\mathbb{E}\Big|\frac{1}{m}\sum_{i=1}^{m}\frac{1}{n}\sum_{j=1}^{n}\partial S_{i,j}(\boldsymbol{\beta})/\partial\boldsymbol{\beta} - \frac{1}{m}\sum_{i=1}^{m}\mathbb{E}\{\partial S_{i,j}(\boldsymbol{\beta})/\partial\boldsymbol{\beta}|X_i\}\Big| \\
\leq&\frac{1}{m}\sum_{i=1}^{m}E\Big|\frac{1}{n}\sum_{j=1}^{n}\partial S_{i,j}(\boldsymbol{\beta})/\partial\boldsymbol{\beta} - \mathbb{E}\{\partial S_{i,j}(\boldsymbol{\beta})/\partial\boldsymbol{\beta}|X_i\}\Big|
\end{aligned}
$$

where the inequality is obtained from $\partial S_{i,j}(\boldsymbol{\beta})/\partial\boldsymbol{\beta}$ are i.i.d. across all j given i. Therefore, we have

$$
P\big\{|\partial S_{m,n}(\boldsymbol{\beta})/\partial\boldsymbol{\beta} - \frac{1}{m}\sum_{i=1}^{m}\mathbb{E}\{\partial S_{i,j}(\boldsymbol{\beta})/\partial\boldsymbol{\beta}|X_i\}| > \epsilon/2\big\} \to 0, \text{ as } \ m,n\to\infty, \tag{S.45}
$$

by weak law of large numbers and convergence in probability is weaker than convergence in mean. With similar arguments, we obtain

$$
P\big\{|\frac{1}{m}\sum_{i=1}^{m}\mathbb{E}\{\partial S_{i,j}(\boldsymbol{\beta})/\partial\boldsymbol{\beta}|X_i\} - \mathbb{E}\{\partial S_{i,j}(\boldsymbol{\beta})/\partial\boldsymbol{\beta}\}| > \epsilon/2\big\} \to 0, \text{ as } \ m,n\to\infty. \tag{S.46}
$$

Combine (S.45) and (S.46), we get

$$
P\big\{|\partial S_{m,n}(\boldsymbol{\beta})/\partial\boldsymbol{\beta} - \mathbb{E}\{\partial S_{i,j}(\boldsymbol{\beta})/\partial\boldsymbol{\beta}\}| > \epsilon\big\} \to 0, \tag{S.47}
$$

as $m,n\to\infty$. Together with Assumption 4, the proof to Condition F3 is complete.

Next, we turn to Condition F2. There exists a union of finite balls with known radius that cover $N_\delta(\boldsymbol{\beta}_0)$. Define balls as $\odot_k$ for all $k \in \{1,\dots,K\}$ with center $\boldsymbol{\beta}_k$ and radius less than r. The finite cover of $N_\delta(\boldsymbol{\beta}_0)$ is $\bigcup_{k=1}^K \odot_k$. By triangle inequality, we have

$$
\begin{aligned}
&\sup_{\boldsymbol{\beta}\in N_\delta(\boldsymbol{\beta}_0)} \Big|\partial S_{m,n}(\boldsymbol{\beta})/\partial\boldsymbol{\beta} - \mathbb{E}\{\partial S_{i,j}(\boldsymbol{\beta})/\partial\boldsymbol{\beta}\}\Big| \\
=&\max_k \sup_{\boldsymbol{\beta}\in\odot_k} \Big|\partial S_{m,n}(\boldsymbol{\beta})/\partial\boldsymbol{\beta} - \partial S_{m,n}(\boldsymbol{\beta}_k)/\partial\boldsymbol{\beta} \\
&\qquad\qquad + \mathbb{E}\{\partial S_{i,j}(\boldsymbol{\beta}_k)/\partial\boldsymbol{\beta}\} - \mathbb{E}\{\partial S_{i,j}(\boldsymbol{\beta})/\partial\boldsymbol{\beta}\} \\
&\qquad\qquad + \partial S_{m,n}(\boldsymbol{\beta}_k)/\partial\boldsymbol{\beta} - \mathbb{E}\{\partial S_{i,j}(\boldsymbol{\beta}_k)/\partial\boldsymbol{\beta}\}\Big| \\
\le&\max_k \sup_{\boldsymbol{\beta}\in\odot_k} \Big|\partial S_{m,n}(\boldsymbol{\beta})/\partial\boldsymbol{\beta} - \partial S_{m,n}(\boldsymbol{\beta}_k)/\partial\boldsymbol{\beta}\Big| \\
&+\max_k \sup_{\boldsymbol{\beta}\in\odot_k} \Big|\mathbb{E}\{\partial S_{i,j}(\boldsymbol{\beta}_k)/\partial\boldsymbol{\beta}\} - \mathbb{E}\{\partial S_{i,j}(\boldsymbol{\beta})/\partial\boldsymbol{\beta}\}\Big| \\
&+\max_k \Big|\partial S_{m,n}(\boldsymbol{\beta}_k)/\partial\boldsymbol{\beta} - \mathbb{E}\{\partial S_{i,j}(\boldsymbol{\beta}_k)/\partial\boldsymbol{\beta}\}\Big| \qquad \text{(S.48)}
\end{aligned}
$$

For the last term in (S.48), we have, for $\epsilon > 0$ and $\tau > 0$,

$$
\begin{aligned}
&P\Big\{\max_k \Big|\partial S_{m,n}(\boldsymbol{\beta}_k)/\partial\boldsymbol{\beta} - \mathbb{E}\{\partial S_{i,j}(\boldsymbol{\beta}_k)/\partial\boldsymbol{\beta}\}\Big| > \epsilon/2\Big\} \\
\le&\sum_{k=1}^K P\Big\{\Big|\partial S_{m,n}(\boldsymbol{\beta}_k)/\partial\boldsymbol{\beta} - \mathbb{E}\{\partial S_{i,j}(\boldsymbol{\beta}_k)/\partial\boldsymbol{\beta}\}\Big| > \epsilon/2\Big\} \\
<&\sum_{k=1}^K \tau/K = \tau, \qquad \text{(S.49)}
\end{aligned}
$$

where the second inequality is obtained from (S.47). For the first term in (S.48), by mean value theorem, we have

$$
\begin{aligned}
&\max_k \sup_{\boldsymbol{\beta}\in\odot_k} \Big|\partial S_{m,n}(\boldsymbol{\beta})/\partial\boldsymbol{\beta} - \partial S_{m,n}(\boldsymbol{\beta}_k)/\partial\boldsymbol{\beta}\Big| \\
=&\max_k \sup_{\boldsymbol{\beta}\in\odot_k} (\boldsymbol{\beta} - \boldsymbol{\beta}_k)\frac{\partial}{\partial\boldsymbol{\beta}}\frac{\partial S_{m,n}(\boldsymbol{\beta}^*)}{\partial\boldsymbol{\beta}} \\
\le& rM_1, \qquad \text{(S.50)}
\end{aligned}
$$

for certain $\beta^* \in \odot_k$. The inequality is obtained from Assumption 3 that derivatives are uniformly bounded by some constant $M_1 = O(1)$. With similar arguments, we have

$$\max_k \sup_{\boldsymbol{\beta} \in \odot_k} \left| \partial S_{m,n}(\boldsymbol{\beta})/\partial \boldsymbol{\beta} - \partial S_{m,n}(\boldsymbol{\beta}_k)/\partial \boldsymbol{\beta} \right| \leq r M_2. \tag{S.51}$$

Make r sufficient small such that $r(M_1 + M_2) \leq \epsilon/2$, combine (S.50), (S.51), (S.49) and (S.48), we have

$$P\left\{ \sup_{\boldsymbol{\beta} \in N_\delta(\boldsymbol{\beta}_0)} \left| \partial S_{m,n}(\boldsymbol{\beta})/\partial \boldsymbol{\beta} - \mathbb{E}\{\partial S_{i,j}(\boldsymbol{\beta})/\partial \boldsymbol{\beta}\} \right| > \epsilon \right\} < \tau. \tag{S.52}$$

The proof of Condition F2 is complete.

Condition F1 is satisfied by Assumption 3 that every term in $\partial S_{m,n}(\boldsymbol{\beta})/\partial \boldsymbol{\beta}$ is at least second-order differentiable. Since $\mathbb{E} V_{i,j}(p_0, q_0) = \mathbb{E} U_{\mathbf{Z}_{i,j}}(p_0, q_0)$, Condition F4 is satisfied. Theorem 2 in (16) can be applied. The proof is complete. ■

S.5 Proof of Theorem 4.4

Our proof strategy is first applying Taylor expansion to get expression $\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0$ by $S_{m,n}(\boldsymbol{\beta}) - S_{m,n}(\boldsymbol{\beta}_0)$, then utilizing a sum to approximate $S_{m,n}(\boldsymbol{\beta}) - S_{m,n}(\boldsymbol{\beta}_0)$, finally prove the limiting distribution of the sum by triangular array central limit theorem.

By Taylor expansion, we get

$$S_{m,n}(\hat{\boldsymbol{\beta}}) - S_{m,n}(\boldsymbol{\beta}_0) \approx (\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0) \frac{\partial S_{m,n}(\boldsymbol{\beta}_0)}{\partial \boldsymbol{\beta}},$$

Hence,

$$\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0 \approx \left(\frac{\partial S_{m,n}(\boldsymbol{\beta}_0)}{\partial \boldsymbol{\beta}} \right)^{-1} \Big(S_{m,n}(\hat{\boldsymbol{\beta}}) - S_{m,n}(\boldsymbol{\beta}_0) \Big).$$

Note that

$$\mathbb{E}\{V_{i,j}(p_0, q_0) | X_i = x_i, \mathbf{Z}_j^{\bar{d}}\} = G_{\mathbf{Z}_j^{\bar{d}}}(x_i),$$

where $G_{\mathbf{Z}_j^{\bar{d}}}(\cdot)$ is the cumulative distribution function of Y conditioned on $\mathbf{Z}_j^{\bar{d}}$. Thus,

$$\mathbb{E}\{G_{\mathbf{Z}_j^{\bar{d}}}(X_i)|\mathbf{Z}_i^d\} = U_{\mathbf{Z}_i^d,\mathbf{Z}_j^{\bar{d}}}(p_0, q_0). \tag{S.53}$$

Similarly, we have

$$\mathbb{E}\{V_{i,j}(p_0, q_0)|Y_j = y_j, \mathbf{Z}_i^d\} = S_{F,\mathbf{Z}_i^d}(y_j),$$

and

$$\mathbb{E}\{S_{F,\mathbf{Z}_i^d}(Y_j)|\mathbf{Z}_j^{\bar{d}}\} = U_{\mathbf{Z}_i^d,\mathbf{Z}_j^{\bar{d}}}(p_0, q_0), \tag{S.54}$$

where $S_{F,\mathbf{Z}_i^d}(\cdot)$ is the survival function of X conditioned on $\mathbf{Z}_i^d$. Then, we get the sum that approximates $S_{m,n}(\boldsymbol{\beta})$,

$$S(\boldsymbol{\beta}) = \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\boldsymbol{\omega}_{i,j}\left[\left(G_{\mathbf{Z}_j^{\bar{d}}}(X_i) - U_{\mathbf{Z}_i^d,\mathbf{Z}_j^{\bar{d}}}(p_0, q_0)\right) + \left(S_{F,\mathbf{Z}_i^d}(Y_j) - U_{\mathbf{Z}_i^d,\mathbf{Z}_j^{\bar{d}}}(p_0, q_0)\right)\right],$$

where

$$\boldsymbol{\omega}_{i,j} = \frac{\partial U_{\mathbf{Z}_i^d,\mathbf{Z}_j^{\bar{d}}}(p_0, q_0)}{\partial \boldsymbol{\beta}}\left(U_{\mathbf{Z}_i^d,\mathbf{Z}_j^{\bar{d}}}(p_0, q_0)\big(1 - U_{\mathbf{Z}_i^d,\mathbf{Z}_j^{\bar{d}}}(p_0, q_0)\big)\right)^{-1}.$$

Combine (S.53) and (S.54), we have

$$S_{m,n}(\boldsymbol{\beta}) - S(\boldsymbol{\beta}) \to 0,$$

in probability, as $m, n \to \infty$. Applying central limit theorem for triangular arrays, we have

$$S(\hat{\boldsymbol{\beta}}) - S(\boldsymbol{\beta}_0) \to N(0, \tilde{\boldsymbol{\Sigma}}),$$

in distribution, where

$$\begin{aligned}\tilde{\boldsymbol{\Sigma}} = &\lim_{m,n\to\infty}\left[\frac{1}{m^2}\sum_{i=1}^{m}\frac{1}{n^2}\sum_{j=1}^{n}\sum_{l=1}^{n}\boldsymbol{\omega}_{i,j}\boldsymbol{\omega}_{i,l}^T\mathrm{Cov}\left(G_{\mathbf{Z}_j^{\bar{d}}}(X_i), G_{\mathbf{Z}_l^{d}}(X_i)\right)\right]\\ &+ \lim_{m,n\to\infty}\left[\frac{1}{n^2}\sum_{j=1}^{n}\frac{1}{m^2}\sum_{i=1}^{m}\sum_{k=1}^{m}\boldsymbol{\omega}_{i,j}\boldsymbol{\omega}_{k,j}^T\mathrm{Cov}\left(S_{F,\mathbf{Z}_i^d}(Y_j), S_{F,\mathbf{Z}_k^d}(Y_j)\right)\right]\end{aligned}$$

Together with (S.52), the proof is complete.

S.6 Additional Simulation

In this section, we present addition simulation results of Section 5 in the main paper.

Case 4: Bootstrap Consistency of the Difference Estimator. We study the coverage probability of 95% confidence interval (4.2) to support the asymptotic normality in Theorem 4.2. Let bootstrap repetition B be 1000. Samples size (m, n) are chosen as: $(80, 80)$, $(150, 150)$ and $(200, 200)$. FPR ($\leq p_0$) and TPR ($\geq q_0$) constraints (p_0, q_0) are $(0.7, 0.5)$, $(0.8, 0.6)$, and $(0.9, 0.7)$. The diseased subjects are generated from $N(\boldsymbol{\mu}_1, \boldsymbol{\Sigma})$ with

$$\boldsymbol{\mu}_1 = (1, 2)^\top \text{ , and } \boldsymbol{\Sigma} = \begin{pmatrix} 1 & 0.8 \\ 0.8 & 1 \end{pmatrix}.$$

The non-diseased are obtained from $N(\boldsymbol{\mu}_2, \boldsymbol{\Sigma})$ with $\boldsymbol{\mu}_2 = (0, 0)^\top$. Simulation in each setting repeats 1000 times. As shown in Table 6, coverage probabilities are around 95%, which supports the asymptotic normality in Theorem 4.2.

Table 6: Coverage probability (CP) of 95% confidence interval (4.2) for the difference estimator $\hat{\theta}(p_0, q_0)$ of two-way (TW) pAUC.

p_0	q_0	m	n	TW pAUC CP
0.7	0.5	50	50	0.948
0.7	0.5	100	100	0.952
0.7	0.5	200	200	0.950
0.8	0.6	50	50	0.952
0.8	0.6	100	100	0.950
0.8	0.6	200	200	0.950
0.9	0.7	50	50	0.949
0.9	0.7	100	100	0.948
0.9	0.7	200	200	0.951

Note: The region of interest is determined by FPR $\leq p_0$ and TPR $\geq q_0$. Coverage probabilities being close to 95% indicates that the asymptotic normality in Theorem 4.2 holds.

Table 7: Coverage probability (CP) of 95% confidence interval (4.1) for $\hat{U}(p_0, q_0)$ with $p_0 = 0.8, q_0 = 0.2$ in data set A, B, and C, respectively.

m	n	CP A	CP B	CP C
30	30	0.919	0.913	0.923
50	50	0.928	0.929	0.932
80	80	0.935	0.933	0.937
100	100	0.937	0.940	0.946
150	150	0.942	0.937	0.945
200	200	0.944	0.946	0.947
150	100	0.957	0.958	0.954
200	150	0.955	0.953	0.951

Note: The region of interest is determined by FPR $\leq p_0$ and TPR $\geq q_0$. The region of interest is determined by FPR $\leq p_0$ and TPR $\geq q_0$. CP being closer to 95% suggests that the asymptotic normality of $\hat{U}(p_0, q_0)$ in Theorem 4.1 holds.

Case 5: Effect of Size on Asymptotic Normality of Estimators. We study the effect of the restricted region's size on the coverage probability of confidence interval (4.1). FPR ($\leq p_0$) and TPR ($\geq q_0$) constraints are $(p_0, q_0) = (0.8, 0.2)$. The rest setup exactly follows Case 1 in Section 5. Combining Table 1 and Table 7, it suggests that larger size of the restricted region ensures higher coverage probability.