Web-based Supplementary Materials for "Polynomial Regression With Heteroscedastic Measurement Errors in Both Axes: Estimation And Hypothesis Testing"

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A Construction of ALS and MALS Estimators

We can construct the variables t_{ri} and h_{ri}, which are needed to set up the ALS and MALS estimating equations for $\boldsymbol{\beta}$, by following the procedure of Cheng and Schneeweiss (1998), which, however, has to be modified in order to allow for heteroscedasticity in the measurement errors. We briefly outline this modified procedure.

The variables t_{ri} are found as the solution to the recursive system

$$x_i^r = \sum_{l=0}^{r} \binom{r}{l} E(\delta_i^{r-l}) t_{li}.$$

The solution for any t_{ri} is a polynomial in x_i of the form

$$t_{ri} = \sum_{l=0}^{r} a_{rli} x_i^l,$$

where the a_{rli} are functions of $E(\delta_i^m)$, $m = 0, \ldots, r$. The h_{ri} are given by

$$h_{ri} = t_{ri} y_i - \sum_{l=0}^{r} b_{rli} t_{li}, \quad b_{rli} = \sum_{s=l}^{r} a_{rsi} \binom{s}{l} E(\delta_i^{s-l} \varepsilon_i).$$

In the important special case when δ_i and ε_i are independent, this reduces to

$$h_{ri} = t_{ri} y_i,$$

because $b_{rli} = 0$ for all r, l, i. Under the assumption of Gaussian errors these expressions simplify considerably. The t_{ri} can be computed recursively (Stulajter, 1978) from

$$t_{r+1,i} = x_i t_{ri} - \sigma_{\delta i}^2 r t_{r-1,i} \quad \text{with} \quad t_{-1i} = t_{0i} = 1.$$

Moreover, it can be shown that

$$h_{ri} = t_{ri} y_i - r \sigma_{\delta\varepsilon i} t_{r-1,i}.$$

With these t_{ri} and h_{ri} the ALS and MALS estimators can be constructed as shown in the main body of the paper. For the cubic model, e.g., we need the t_{ri} up to $r = 6$ and the h_{ri} up to $r = 3$. In the main body of the paper these expressions are displayed for normally distributed measurement errors. We repeat them for the convenience of the reader.

$$t_{0i} = 1, \quad t_{1i} = x_i, \quad t_{2i} = x_i^2 - \sigma_{\delta i}^2, \quad t_{3i} = x_i^3 - 3x_i\sigma_{\delta i}^2,$$

$$t_{4i} = x_i^4 - 6x_i^2\sigma_{\delta i}^2 + 3\sigma_{\delta i}^4, \quad t_{5i} = x_i^5 - 10x_i^3\sigma_{\delta i}^2 + 15x_i\sigma_{\delta i}^4,$$

$$t_{6i} = x_i^6 - 15x_i^4\sigma_{\delta i}^2 + 45x_i^2\sigma_{\delta i}^4 - 15\sigma_{\delta i}^6.$$

$$h_{0i} = y_i, \quad h_{1i} = t_{1i}y_i - \sigma_{\delta\varepsilon i}, \quad h_{2i} = t_{2i}y_i - 2\sigma_{\delta\varepsilon i}t_{1i}, \quad h_{3i} = t_{3i}y_i - 3\sigma_{\delta\varepsilon i}t_{2i}.$$

For setting up MALS estimating equations for a cubic model we first need to compute the statistic γ, see the main part of the paper. Suppose this has been done, the MALS estimating equations are then given as follows, where the u_{ri} are implicitly defined. (For $\gamma = 1$ they turn into the ALS estimating equations).

$$\sum_{i=1}^n u_{0i} = \sum_{i=1}^n (y_i - \hat\beta_0 - t_{1i}\hat\beta_1 - t_{2i}\hat\beta_2 - t_{3i}\hat\beta_3) = 0, \tag{1}$$

$$\begin{aligned}\sum_{i=1}^n u_{1i} = \sum_{i=1}^n \Big[&(y_i t_{1i} - \gamma\sigma_{\delta\varepsilon i}) - t_{1i}\hat\beta_0 \\ &- \{\gamma t_{2i} + (1-\gamma)t_{1i}^2\}\hat\beta_1 \\ &- \{\gamma t_{3i} + (1-\gamma)t_{1i}t_{2i}\}\hat\beta_2 \\ &- \{\gamma t_{4i} + (1-\gamma)t_{1i}t_{3i}\}\hat\beta_3\Big] = 0,\end{aligned} \tag{2}$$

$$\begin{aligned}\sum_{i=1}^n u_{2i} = \sum_{i=1}^n \Big[&(y_i t_{2i} - 2\gamma\sigma_{\delta\varepsilon i}t_{1i}) - t_{2i}\hat\beta_0 \\ &- \{\gamma t_{3i} + (1-\gamma)t_{1i}t_{2i}\}\hat\beta_1 \\ &- \{\gamma t_{4i} + (1-\gamma)t_{2i}^2\}\hat\beta_2 \\ &- \{\gamma t_{5i} + (1-\gamma)t_{2i}t_{3i}\}\hat\beta_3\Big] = 0,\end{aligned} \tag{3}$$

$$\begin{aligned}\sum_{i=1}^n u_{3i} = \sum_{i=1}^n \Big[&(y_i t_{3i} - 3\gamma\sigma_{\delta\varepsilon i}t_{2i}) - t_{3i}\hat\beta_0 \\ &- \{\gamma t_{4i} + (1-\gamma)t_{1i}t_{3i}\}\hat\beta_1 \\ &- \{\gamma t_{5i} + (1-\gamma)t_{2i}t_{3i}\}\hat\beta_2 \\ &- \{\gamma t_{6i} + (1-\gamma)t_{3i}^2\}\hat\beta_3\Big] = 0.\end{aligned} \tag{4}$$

The asymptotic covariance matrix of $\hat{\boldsymbol{\beta}} = (\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2, \hat{\beta}_3)$ can be estimated by $n^{-1}\boldsymbol{S}^{-1}\boldsymbol{U}\boldsymbol{S}^{-1}$, where $\boldsymbol{S}$ and $\boldsymbol{U}$ are 4 by 4 symmetric matrices with elements s_{lm} and u_{lm}, respectively, defined by

$$s_{00} = 1, \quad s_{01} = n^{-1}\sum_{i=1}^{n} t_{1i}, \quad s_{02} = n^{-1}\sum_{i=1}^{n} t_{2i},$$
$$s_{03} = n^{-1}\sum_{i=1}^{n} t_{3i}, \quad s_{11} = n^{-1}\sum_{i=1}^{n}\{\gamma t_{2i} + (1-\gamma)t_{1i}^2\},$$
$$s_{12} = n^{-1}\sum_{i=1}^{n}\{\gamma t_{3i} + (1-\gamma)t_{1i}t_{2i}\},$$
$$s_{13} = n^{-1}\sum_{i=1}^{n}\{\gamma t_{4i} + (1-\gamma)t_{1i}t_{3i}\},$$
$$s_{22} = n^{-1}\sum_{i=1}^{n}\{\gamma t_{4i} + (1-\gamma)t_{2i}^2\},$$
$$s_{23} = n^{-1}\sum_{i=1}^{n}\{\gamma t_{5i} + (1-\gamma)t_{2i}t_{3i}\},$$
$$s_{33} = n^{-1}\sum_{i=1}^{n}\{\gamma t_{6i} + (1-\gamma)t_{3i}^2\},$$

and

$$u_{lm} = n^{-1}\sum_{i=1}^{n} u_{li}u_{mi}.$$

For the quadratic model, simply let $\hat{\beta}_3$ equal zero in the first three estimating equations of the cubic model and delete the fourth equation.

B Simulations Studies and Methods Comparisons

The following results of a simulation study extend those of the main body of the paper, where the parameter settings are described in full detail. The regression parameter for quadratic model is $\boldsymbol{\beta}' = (0.25, 1, 1)$ and for the cubic model is $\boldsymbol{\beta}' = (0.25, 1, 1, -1)$, respectively.

Tables 1 and 2 are similar to Table 1 in the main part, but the cases dealt with have been extended to small and large sample sizes ($n = 10, 20, 200$). It is remarkable that even for very small sample sizes (like $n = 10$) MALS and MCS give satisfactory results, for CS and MCS see Zavala et al. (2007). Only σ_q^2 is estimated with a sizable (positive) bias if $n = 10$.

In Table 3 and 4 we compare ALS with the MALS and MCS methods when $\sigma_q^2 = 1$. As MALS1 and MALS2 are about the same in this case, we present a comparison of ALS to MCS and we only show it for the quadratic model. In Tables 3 and 4 (and also in Table 6), the simulation set up deviates somewhat from the general set up as described in the main body of the paper in so far as $\overline{\sigma_\delta^2}$ is not kept fixed. Instead several values of $\overline{\sigma_\delta^2}$ have been chosen, ranging from 0.01 to 0.1. The corresponding values of κ_a are shown in the table. In this way we can study the effect of different values of the reliability ratio on the performance of the various estimators.

Table 1: Estimates (standard errors) of MALS1, MALS2 and MCS for the quadratic model. ($\kappa_a = 0.77$)

		MALS1				MCS		
σ_q^2	n	$\hat\beta_0$	$\hat\beta_1$	$\hat\beta_2$	$\hat\sigma_q^2$	$\hat\beta_0$	$\hat\beta_1$	$\hat\beta_2$
1	10	0.30 (0.66)	1.04 (0.66)	0.99 (1.42)	1.37	0.30 (0.66)	1.04 (0.67)	0.98 (1.44)
	20	0.18 (0.73)	1.06 (0.62)	1.27 (1.70)	0.97	0.19 (0.75)	1.06 (0.65)	1.26 (1.76)
	30	0.13 (0.70)	1.06 (0.54)	1.37 (1.66)	0.91	0.12 (0.73)	1.07 (0.56)	1.38 (1.74)
	50	0.11 (0.59)	1.04 (0.42)	1.41 (1.41)	0.90	0.10 (0.62)	1.05 (0.44)	1.43 (1.50)
	100	0.15 (0.36)	1.02 (0.26)	1.27 (0.84)	0.94	0.14 (0.38)	1.03 (0.28)	1.30 (0.90)
	200	0.21 (0.21)	1.01 (0.17)	1.11 (0.49)	0.97	0.21 (0.22)	1.01 (0.18)	1.12 (0.52)
0	10	0.33 (0.31)	1.00 (0.33)	0.91 (0.67)	0.21	0.33 (0.31)	1.00 (0.33)	0.90 (0.67)
	20	0.26 (0.34)	1.02 (0.31)	1.06 (0.77)	0.03	0.26 (0.34)	1.02 (0.32)	1.04 (0.77)
	30	0.21 (0.33)	1.02 (0.28)	1.16 (0.78)	-0.00	0.21 (0.33)	1.02 (0.29)	1.15 (0.79)
	50	0.19 (0.29)	1.02 (0.23)	1.20 (0.69)	-0.02	0.19 (0.29)	1.02 (0.23)	1.20 (0.70)
	100	0.19 (0.21)	1.02 (0.16)	1.18 (0.52)	-0.02	0.19 (0.22)	1.02 (0.16)	1.18 (0.53)
	200	0.21 (0.14)	1.01 (0.11)	1.11 (0.35)	-0.02	0.21 (0.15)	1.01 (0.11)	1.12 (0.36)

		MALS2			
σ_q^2	n	$\hat\beta_0$	$\hat\beta_1$	$\hat\beta_2$	$\hat\sigma_q^2$
1	10	0.31 (0.64)	1.03 (0.65)	0.96 (1.38)	1.44
	20	0.20 (0.70)	1.05 (0.61)	1.22 (1.63)	0.99
	30	0.14 (0.67)	1.06 (0.53)	1.33 (1.59)	0.92
	50	0.12 (0.57)	1.04 (0.41)	1.38 (1.36)	0.91
	100	0.15 (0.35)	1.02 (0.26)	1.27 (0.83)	0.94
	200	0.21 (0.21)	1.01 (0.17)	1.11 (0.49)	0.97
0	10	0.36 (0.28)	0.97 (0.30)	0.82 (0.57)	0.39
	20	0.33 (0.26)	0.98 (0.26)	0.87 (0.57)	0.16
	30	0.30 (0.24)	0.98 (0.23)	0.92 (0.54)	0.10
	50	0.28 (0.20)	0.99 (0.18)	0.95 (0.46)	0.06
	100	0.27 (0.15)	0.99 (0.14)	0.96 (0.36)	0.04
	200	0.27 (0.12)	0.99 (0.10)	0.97 (0.28)	0.03

From Table 3 it can be seen that ALS is unstable when the sample size is small even when the average reliability ratio is as high as 0.82. However, its modified version MALS1 can significantly reduce its bias and standard

Table 2: Estimates (standard errors) of MALS1, MALS2 and MCS for the cubic model. ($\kappa_a = 0.93$)

		MALS1					MCS			
σ_q^2	n	$\hat\beta_0$	$\hat\beta_1$	$\hat\beta_2$	$\hat\beta_3$	$\hat\sigma_q^2$	$\hat\beta_0$	$\hat\beta_1$	$\hat\beta_2$	$\hat\beta_3$
3	10	0.31 (0.90)	0.93 (1.53)	1.03 (0.51)	-1.08 (0.55)	7.64	0.31 (0.90)	0.92 (1.53)	1.03 (0.51)	-1.08 (0.55)
	20	0.25 (0.82)	1.30 (1.68)	1.04 (0.50)	-1.17 (0.58)	3.55	0.25 (0.84)	1.29 (1.73)	1.04 (0.52)	-1.17 (0.60)
	30	0.22 (0.74)	1.47 (1.64)	1.04 (0.47)	-1.22 (0.58)	2.83	0.22 (0.77)	1.47 (1.71)	1.05 (0.48)	-1.22 (0.60)
	50	0.19 (0.62)	1.59 (1.48)	1.05 (0.40)	-1.25 (0.53)	2.46	0.19 (0.64)	1.60 (1.55)	1.06 (0.41)	-1.26 (0.56)
	100	0.22 (0.45)	1.51 (1.13)	1.03 (0.30)	-1.20 (0.41)	2.46	0.22 (0.47)	1.53 (1.18)	1.03 (0.31)	-1.21 (0.43)
	200	0.22 (0.32)	1.32 (0.82)	1.02 (0.22)	-1.13 (0.31)	2.65	0.22 (0.34)	1.35 (0.86)	1.02 (0.23)	-1.14 (0.32)
0	10	0.36 (0.44)	0.65 (0.82)	1.00 (0.32)	-0.96 (0.31)	4.48	0.36 (0.44)	0.65 (0.82)	1.00 (0.32)	-0.96 (0.31)
	20	0.37 (0.39)	0.71 (0.87)	0.96 (0.30)	-0.94 (0.30)	1.55	0.36 (0.40)	0.70 (0.89)	0.96 (0.31)	-0.94 (0.32)
	30	0.34 (0.36)	0.78 (0.83)	0.97 (0.28)	-0.95 (0.30)	0.96	0.34 (0.37)	0.78 (0.86)	0.97 (0.29)	-0.95 (0.30)
	50	0.33 (0.31)	0.85 (0.77)	0.97 (0.25)	-0.96 (0.28)	0.59	0.33 (0.32)	0.85 (0.79)	0.97 (0.26)	-0.96 (0.29)
	100	0.30 (0.26)	0.88 (0.67)	0.97 (0.21)	-0.97 (0.25)	0.34	0.30 (0.26)	0.88 (0.68)	0.97 (0.21)	-0.97 (0.26)
	200	0.29 (0.21)	0.90 (0.57)	0.98 (0.17)	-0.97 (0.22)	0.22	0.29 (0.21)	0.89 (0.58)	0.98 (0.17)	-0.97 (0.22)

		MALS2				
σ_q^2	n	$\hat\beta_0$	$\hat\beta_1$	$\hat\beta_2$	$\hat\beta_3$	$\hat\sigma_q^2$
3	10	0.31 (0.90)	0.91 (1.52)	1.03 (0.51)	-1.07 (0.54)	7.83
	20	0.26 (0.81)	1.27 (1.65)	1.03 (0.50)	-1.16 (0.58)	3.66
	30	0.23 (0.73)	1.44 (1.61)	1.04 (0.47)	-1.21 (0.57)	2.92
	50	0.20 (0.61)	1.55 (1.45)	1.05 (0.40)	-1.24 (0.52)	2.53
	100	0.22 (0.45)	1.49 (1.12)	1.03 (0.30)	-1.20 (0.41)	2.49
	200	0.23 (0.32)	1.31 (0.82)	1.02 (0.22)	-1.12 (0.30)	2.66
0	10	0.36 (0.44)	0.64 (0.81)	1.00 (0.32)	-0.96 (0.31)	4.75
	20	0.37 (0.38)	0.67 (0.85)	0.96 (0.29)	-0.93 (0.30)	1.73
	30	0.34 (0.35)	0.73 (0.80)	0.97 (0.28)	-0.94 (0.28)	1.13
	50	0.34 (0.31)	0.78 (0.74)	0.96 (0.25)	-0.94 (0.27)	0.74
	100	0.31 (0.25)	0.81 (0.64)	0.97 (0.20)	-0.94 (0.24)	0.48
	200	0.30 (0.20)	0.83 (0.55)	0.97 (0.16)	-0.94 (0.21)	0.34

deviation for samples as small as 10. This implies that the modification of ALS is necessary when the sample size is small and/or the average reliability ratio is not very high. The comparison of CS versus MCS is similar and hence is omitted. The reason is that the modification of CS follows the same line as the modification of ALS.

If $\sigma_{\varepsilon i}^2$ was chosen to lie between 0.001 and 0.002, the standard deviations of the $\hat\beta$'s were larger for MCS than for MALS1 or MALS2 even if $\sigma_q^2 = 0$ (see Table 5 for the reliability ratio is 0.77 case). ACS performs very poorly in this situation, it has very large standard deviation because CS can be modified into MCS but ACS being computed by an algorithm, it is not clear that how can it be modified into MACS. One reason of the relative poor performance of MCS, especially ACS, in this case is the fact that $\sigma_{\varepsilon i}^2$ can become very small yielding rather unstable results for WLS (weighted least squares), which underlies MCS.

In Table 6 we introduce naïve OLS and ACS in our comparison, for ACS

Table 3: Estimates (standard errors) of ALS and MALS1 for the quadratic model with equation error ($\sigma_q^2 = 1$).

		ALS				MALS1			
κ_a	n	$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\sigma}_q^2$	$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\sigma}_q^2$
0.97	10	0.22 (0.47)	1.01 (0.41)	1.09 (0.80)	0.60	0.24 (0.45)	1.00 (0.41)	1.04 (0.76)	0.97
	20	0.24 (0.36)	1.01 (0.34)	1.05 (0.65)	0.80	0.25 (0.35)	1.01 (0.34)	1.02 (0.63)	0.98
	30	0.24 (0.30)	1.01 (0.30)	1.03 (0.55)	0.87	0.24 (0.30)	1.00 (0.30)	1.02 (0.55)	0.99
	40	0.24 (0.26)	1.00 (0.26)	1.01 (0.49)	0.90	0.25 (0.26)	1.00 (0.26)	1.00 (0.48)	1.00
	50	0.24 (0.24)	1.01 (0.24)	1.02 (0.44)	0.92	0.25 (0.24)	1.00 (0.24)	1.02 (0.44)	0.99
	80	0.25 (0.19)	1.01 (0.19)	1.02 (0.36)	0.95	0.25 (0.19)	1.01 (0.19)	1.01 (0.36)	1.00
	100	0.24 (0.17)	1.00 (0.17)	1.01 (0.32)	0.96	0.24 (0.17)	1.00 (0.17)	1.01 (0.32)	0.99
	200	0.25 (0.12)	1.00 (0.12)	1.01 (0.23)	0.98	0.25 (0.12)	1.00 (0.12)	1.00 (0.23)	1.00
0.91	10	0.10 (27.29)	1.07 (18.82)	1.55 (94.59)	0.48	0.21 (0.55)	1.03 (0.49)	1.16 (1.05)	0.94
	20	0.18 (0.64)	1.03 (0.41)	1.23 (1.35)	0.75	0.22 (0.42)	1.02 (0.38)	1.12 (0.82)	0.95
	30	0.20 (0.36)	1.02 (0.33)	1.14 (0.74)	0.83	0.23 (0.35)	1.01 (0.32)	1.08 (0.68)	0.96
	40	0.21 (0.31)	1.02 (0.29)	1.12 (0.63)	0.87	0.22 (0.30)	1.01 (0.28)	1.08 (0.60)	0.97
	50	0.22 (0.28)	1.02 (0.26)	1.09 (0.56)	0.90	0.23 (0.27)	1.01 (0.26)	1.05 (0.54)	0.98
	80	0.23 (0.22)	1.01 (0.21)	1.06 (0.44)	0.94	0.24 (0.22)	1.00 (0.21)	1.04 (0.43)	0.98
	100	0.24 (0.19)	1.01 (0.19)	1.04 (0.39)	0.95	0.24 (0.19)	1.00 (0.19)	1.03 (0.39)	0.99
	200	0.24 (0.14)	1.00 (0.13)	1.02 (0.28)	0.98	0.25 (0.14)	1.00 (0.13)	1.01 (0.28)	1.00
0.87	10	0.08 (495.41)	1.20 (221.51)	1.60 (1356.56)	0.55	0.20 (0.59)	1.04 (0.53)	1.22 (1.20)	0.98
	20	0.11 (36.26)	1.08 (8.13)	1.39 (110.09)	0.67	0.19 (0.49)	1.03 (0.42)	1.19 (1.02)	0.92
	30	0.09 (3.72)	1.01 (2.21)	1.47 (12.08)	0.76	0.19 (0.40)	1.02 (0.36)	1.18 (0.84)	0.94
	40	0.17 (0.40)	1.03 (0.34)	1.23 (0.89)	0.85	0.21 (0.34)	1.02 (0.31)	1.14 (0.72)	0.95
	50	0.19 (0.32)	1.02 (0.28)	1.18 (0.67)	0.88	0.21 (0.30)	1.02 (0.28)	1.11 (0.63)	0.96
	80	0.21 (0.24)	1.02 (0.22)	1.11 (0.51)	0.92	0.22 (0.24)	1.01 (0.22)	1.07 (0.49)	0.97
	100	0.22 (0.22)	1.01 (0.20)	1.10 (0.46)	0.94	0.23 (0.21)	1.01 (0.20)	1.07 (0.44)	0.98
	200	0.24 (0.15)	1.01 (0.14)	1.03 (0.32)	0.97	0.25 (0.15)	1.00 (0.14)	1.02 (0.31)	0.99
0.83	10	0.03 (258.86)	1.33 (200.41)	1.27 (768.70)	0.11	0.25 (0.63)	1.04 (0.59)	1.10 (1.31)	1.12
	20	0.77 (957.82)	1.23 (238.13)	0.37 (2499.25)	0.62	0.16 (0.59)	1.06 (0.50)	1.30 (1.31)	0.91
	30	0.20 (54.50)	1.07 (11.79)	1.17 (144.15)	0.77	0.16 (0.51)	1.02 (0.41)	1.28 (1.15)	0.91
	40	0.15 (6.52)	1.02 (3.09)	1.29 (17.38)	0.83	0.16 (0.44)	1.03 (0.36)	1.26 (0.98)	0.93
	50	0.09 (0.96)	1.04 (0.47)	1.41 (2.59)	0.83	0.16 (0.38)	1.02 (0.31)	1.23 (0.84)	0.93
	80	0.18 (0.30)	1.02 (0.25)	1.19 (0.66)	0.91	0.21 (0.28)	1.02 (0.24)	1.13 (0.61)	0.97
	100	0.20 (0.26)	1.02 (0.22)	1.15 (0.57)	0.93	0.22 (0.24)	1.02 (0.21)	1.10 (0.54)	0.97
	200	0.22 (0.17)	1.01 (0.15)	1.07 (0.38)	0.97	0.23 (0.17)	1.00 (0.15)	1.05 (0.37)	0.99
0.77	10	58.81 ($> 10^5$)	85.68 ($> 10^5$)	-166.43 ($> 10^5$)	32.98	0.31 (0.64)	1.05 (0.67)	1.00 (1.41)	1.40
	20	2.02 ($> 10^5$)	0.55 (5699.57)	-4.35 ($> 10^5$)	1.12	0.20 (0.71)	1.05 (0.61)	1.20 (1.63)	1.00
	30	0.36 (413.81)	1.49 (214.29)	1.19 (1141.19)	0.78	0.13 (0.69)	1.07 (0.53)	1.35 (1.66)	0.90
	40	0.19 (129.08)	1.02 (69.77)	1.14 (368.56)	0.78	0.11 (0.64)	1.05 (0.47)	1.43 (1.54)	0.89
	50	-0.54 (448.79)	1.29 (104.07)	3.07 (1179.61)	0.61	0.10 (0.59)	1.04 (0.42)	1.44 (1.43)	0.90
	80	0.06 (7.75)	1.07 (4.28)	1.50 (20.69)	0.85	0.13 (0.43)	1.03 (0.30)	1.34 (1.03)	0.93
	100	0.08 (5.67)	1.03 (0.50)	1.47 (16.21)	0.88	0.16 (0.35)	1.03 (0.26)	1.26 (0.82)	0.94
	200	0.19 (0.22)	1.02 (0.17)	1.15 (0.52)	0.95	0.21 (0.21)	1.01 (0.17)	1.12 (0.50)	0.97

see Patriota and Bolfarine (2008). Generally, OLS has a larger bias but a smaller standard deviation than ALS and ACS. Intuitively, when the average reliability ratio is very high, i.e., when the measurement error is very small, OLS should perform well because the model is close to an ordinary regression model. But even in the case of a large reliability ratio, OLS shows inconsistency. Simulations also indicate that ACS is about the same as ALS (see Tables 3 and 6). However, when the sample size is not large, such as smaller than 40, ACS fluctuates greatly, especially when the average reliability ratio is below 0.77. ALS can be modified into MALS but it is not clear how one can modify ACS because ACS is produced by an iterative numerical algorithm.

Table 7 and 8 display the simulations of rejection rates of Wald-type and

Table 4: Estimates (standard errors) of MALS2 and MCS for the quadratic model with equation error ($\sigma_q^2 = 1$).

		MALS2				MCS		
κ_a	n	$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\sigma}_q^2$	$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\beta}_2$
0.97	10	0.24 (0.45)	1.00 (0.41)	1.05 (0.75)	0.97	0.23 (0.45)	1.00 (0.41)	1.05 (0.76)
	20	0.24 (0.35)	1.01 (0.34)	1.02 (0.63)	0.98	0.24 (0.36)	1.01 (0.34)	1.02 (0.64)
	30	0.24 (0.30)	1.00 (0.29)	1.01 (0.55)	0.99	0.24 (0.30)	1.00 (0.30)	1.01 (0.55)
	40	0.24 (0.26)	1.00 (0.26)	1.02 (0.48)	0.99	0.24 (0.27)	1.00 (0.26)	1.02 (0.49)
	50	0.24 (0.24)	1.00 (0.24)	1.02 (0.44)	0.99	0.24 (0.24)	1.00 (0.24)	1.02 (0.45)
	80	0.25 (0.19)	1.00 (0.19)	1.01 (0.36)	1.00	0.25 (0.20)	1.00 (0.20)	1.01 (0.36)
	100	0.25 (0.17)	1.00 (0.17)	1.01 (0.32)	1.00	0.25 (0.18)	1.00 (0.18)	1.01 (0.33)
	200	0.25 (0.12)	1.00 (0.12)	1.00 (0.23)	1.00	0.25 (0.13)	1.00 (0.13)	1.00 (0.24)
0.91	10	0.21 (0.54)	1.04 (0.48)	1.16 (1.03)	0.98	0.20 (0.55)	1.05 (0.49)	1.20 (1.08)
	20	0.20 (0.42)	1.03 (0.38)	1.15 (0.82)	0.94	0.20 (0.43)	1.03 (0.39)	1.16 (0.85)
	30	0.22 (0.35)	1.01 (0.32)	1.09 (0.69)	0.97	0.21 (0.36)	1.01 (0.33)	1.11 (0.71)
	40	0.23 (0.30)	1.01 (0.28)	1.06 (0.60)	0.97	0.23 (0.31)	1.02 (0.29)	1.06 (0.61)
	50	0.23 (0.27)	1.00 (0.26)	1.06 (0.54)	0.98	0.23 (0.28)	1.00 (0.26)	1.06 (0.55)
	100	0.24 (0.19)	1.01 (0.19)	1.03 (0.39)	0.99	0.24 (0.20)	1.01 (0.19)	1.03 (0.40)
	200	0.24 (0.14)	1.00 (0.13)	1.02 (0.28)	0.99	0.24 (0.14)	1.00)0.14)	1.02 (0.29)
0.87	10	0.22 (0.59)	1.04 (0.53)	1.15 (1.20)	1.05	0.20 (0.61)	1.05 (0.55)	1.20 (1.26)
	20	0.19 (0.49)	1.03 (0.43)	1.20 (1.01)	0.93	0.18 (0.51)	1.03 (0.44)	1.22 (1.06)
	30	0.20 (0.40)	1.01 (0.35)	1.17 (0.84)	0.95	0.19 (0.42)	1.01 (0.36)	1.20 (0.88)
	40	0.21 (0.34)	1.01 (0.31)	1.12 (0.71)	0.96	0.20 (0.35)	1.02 (0.31)	1.14 (0.74)
	50	0.21 (0.30)	1.02 (0.28)	1.12 (0.63)	0.97	0.21 (0.31)	1.02 (0.28)	1.13 (0.65)
	80	0.23 (0.24)	1.00 (0.22)	1.06 (0.49)	0.98	0.23 (0.24)	1.00 (0.22)	1.06 (0.51)
	100	0.23 (0.21)	1.01 (0.20)	1.05 (0.44)	0.98	0.23 (0.22)	1.01 (0.20)	1.06 (0.46)
	200	0.24 (0.15)	1.00 (0.14)	1.02 (0.31)	0.99	0.24 (0.15)	1.00 (0.14)	1.03 (0.32)
0.83	10	0.24 (0.62)	1.02 (0.58)	1.13 (1.29)	1.18	0.22 (0.64)	1.03 (0.60)	1.16 (1.36)
	20	0.17 (0.58)	1.05 (0.50)	1.28 (1.29)	0.92	0.16 (0.62)	1.06 (0.52)	1.32 (1.39)
	30	0.16 (0.50)	1.04 (0.41)	1.27 (1.11)	0.91	0.15 (0.53)	1.04 (0.43)	1.31 (1.19)
	40	0.17 (0.43)	1.03 (0.35)	1.23 (0.95)	0.93	0.16 (0.46)	1.04 (0.37)	1.25 (1.02)
	50	0.18 (0.37)	1.02 (0.31)	1.20 (0.82)	0.94	0.17 (0.40)	1.02 (0.33)	1.23 (0.88)
	80	0.21 (0.28)	1.02 (0.24)	1.11 (0.60)	0.96	0.21 (0.29)	1.02 (0.25)	1.13 (0.63)
	100	0.21 (0.25)	1.01 (0.21)	1.10 (0.54)	0.98	0.21 (0.26)	1.01 (0.22)	1.11 (0.56)
	200	0.23 (0.17)	1.00 (0.15)	1.06 (0.37)	0.98	0.23 (0.18)	1.00 (0.16)	1.07 (0.38)
0.77	10	0.31 (0.63)	1.03 (0.63)	0.96 (1.33)	1.42	0.30 (0.64)	1.04 (0.65)	0.99 (1.39)
	20	0.20 (0.68)	1.06 (0.60)	1.22 (1.60)	0.99	0.19 (0.73)	1.06 (0.63)	1.26 (1.72)
	30	0.14 (0.66)	1.06 (0.53)	1.33 (1.55)	0.92	0.13 (0.71)	1.06 (0.56)	1.37 (1.70)
	40	0.14 (0.60)	1.05 (0.45)	1.35 (1.41)	0.91	0.12 (0.65)	1.06 (0.49)	1.40 (1.56)
	50	0.13 (0.55)	1.05 (0.40)	1.35 (1.31)	0.92	0.11 (0.61)	1.05 (0.43)	1.42 (1.45)
	80	0.15 (0.41)	1.03 (0.30)	1.28 (0.98)	0.94	0.13 (0.45)	1.03 (0.32)	1.32 (1.08)
	100	0.16 (0.35)	1.02 (0.26)	1.25 (0.82)	0.94	0.15 (0.38)	1.02 (0.27)	1.28 (0.88)
	200	0.21 (0.21)	1.01 (0.17)	1.12 (0.50)	0.97	0.20 (0.23)	1.01 (0.18)	1.14 (0.52)

score-type tests induced by MALS and MCS for $k = 2$ and $k = 3$, respectively. They are similar to Table 2 in the main part, but the cases dealt with have been extended to small and large sample sizes ($n = 10, 20, 200$).

An interesting observation comparing Wald-type to score-type tests is that the latter performs better in small to moderate samples in the sense that the rejection rate is closer to the nominal value under the null-hypothesis. This trend was also reported in Gimenez et al. (2000, p. 703) for Wald-type and score-type tests based on corrected score methodology. But when the sample is large, the difference between these two tests diminishes.

In summary, the MCS estimator is not suitable for models with an equation error. When the model is without an equation error, MALS1 and MCS are very close and MALS2 is often better than MALS1. The same conclusion applies to

Table 5: Estimates (standard errors) of ALS, CS, MALS1, MALS2 and MCS for the quadratic model. ($\kappa_a = 0.77$ and taking $\sigma^2_{\varepsilon i}$ between 0.001 and 0.002)

		ALS				CS		
σ^2_q	n	$\hat\beta_0$	$\hat\beta_1$	$\hat\beta_2$	$\hat\sigma^2_q$	$\hat\beta_0$	$\hat\beta_1$	$\hat\beta_2$
1	10	7.44 (14120.91)	0.64 (2064.92)	-19.52 (40614.62)	2.95	-1.62 (6906.44)	1.03 (2322.50)	3.00 (13197.20)
	20	0.77 (178.05)	2.26 (474.88)	-0.89 (771.11)	0.83	0.52 (186.97)	0.98 (60.57)	0.77 (603.67)
	30	0.27 (62.40)	0.93 (13.16)	1.13 (167.45)	0.67	-0.81 (92.10)	1.48 (39.10)	3.74 (251.84)
	40	0.20 (92.71)	0.88 (100.83)	1.27 (189.86)	0.59	-0.53 (187.54)	1.20 (31.47)	3.00 (457.10)
	50	0.29 (21.22)	1.01 (7.94)	0.81 (58.98)	0.76	-0.02 (47.86)	0.98 (4.15)	1.68 (139.56)
	80	0.15 (6.41)	1.03 (1.85)	1.33 (17.72)	0.68	0.45 (42.93)	1.08 (2.82)	1.46 (114.22)
	100	0.19 (1.75)	1.01 (0.67)	1.17 (4.44)	0.73	0.11 (0.59)	1.03 (0.34)	1.37 (1.46)
	200	0.19 (0.21)	1.01 (0.16)	1.16 (0.49)	0.75	0.19 (0.21)	1.02 (0.17)	1.16 (0.50)
0	10	0.56 (144.88)	0.79 (125.07)	0.22 (380.79)	-0.13	1.39 (735.09)	0.04 (654.39)	-2.04 (1743.72)
	20	-0.23 (286.66)	1.20 (106.00)	2.32 (628.62)	-0.40	0.24 (93.26)	1.04 (63.55)	0.95 (226.33)
	30	-0.41 (246.88)	1.04 (497.00)	2.76 (357.41)	-0.28	-0.45 (144.00)	1.34 (73.07)	3.11 (438.87)
	40	0.20 (13.51)	1.19 (8.63)	1.05 (37.05)	-0.25	0.28 (38.83)	1.05 (13.12)	0.95 (101.14)
	50	0.20 (40.51)	1.06 (14.00)	1.12 (112.44)	-0.22	0.22 (32.12)	1.04 (4.72)	1.11 (86.25)
	80	0.16 (3.50)	1.05 (1.34)	1.25 (9.47)	-0.24	0.08 (2.29)	1.03 (0.65)	1.45 (6.30)
	100	0.17 (1.80)	1.02 (0.71)	1.27 (5.09)	-0.23	0.11 (0.59)	1.05 (0.28)	1.40 (1.57)
	200	0.20 (0.14)	1.02 (0.10)	1.14 (0.35)	-0.22	0.20 (0.14)	1.02 (0.10)	1.15 (0.36)

		MALS1				MCS		
σ^2_q	n	$\hat\beta_0$	$\hat\beta_1$	$\hat\beta_2$	$\hat\sigma^2_q$	$\hat\beta_0$	$\hat\beta_1$	$\hat\beta_2$
1	10	0.32 (0.59)	1.04 (0.60)	0.93 (1.27)	1.32	0.32 (0.59)	1.04 (0.60)	0.94 (1.27)
	20	0.19 (0.65)	1.09 (0.58)	1.23 (1.52)	0.95	0.19 (0.66)	1.09 (0.59)	1.22 (1.55)
	30	0.13 (0.62)	1.05 (0.49)	1.36 (1.47)	0.92	0.13 (0.63)	1.06 (0.50)	1.36 (1.50)
	40	0.12 (0.58)	1.06 (0.43)	1.36 (1.36)	0.90	0.12 (0.59)	1.06 (0.44)	1.36 (1.40)
	50	0.12 (0.53)	1.02 (0.39)	1.35 (1.28)	0.94	0.13 (0.55)	1.02 (0.39)	1.35 (1.31)
	80	0.16 (0.40)	1.03 (0.29)	1.30 (0.97)	0.92	0.15 (0.41)	1.03 (0.30)	1.31 (0.99)
	100	0.18 (0.32)	1.01 (0.24)	1.19 (0.75)	0.95	0.18 (0.33)	1.02 (0.25)	1.20 (0.77)
	200	0.21 (0.20)	1.01 (0.16)	1.12 (0.46)	0.97	0.21 (0.21)	1.01 (0.16)	1.13 (0.47)
0	10	0.36 (0.18)	0.97 (0.21)	0.81 (0.37)	0.20	0.37 (0.18)	0.97 (0.21)	0.81 (0.38)
	20	0.33 (0.18)	0.97 (0.19)	0.85 (0.39)	0.09	0.33 (0.18)	0.97 (0.19)	0.85 (0.40)
	30	0.32 (0.17)	0.98 (0.17)	0.88 (0.38)	0.06	0.32 (0.17)	0.98 (0.17)	0.88 (0.38)
	40	0.31 (0.16)	0.97 (0.15)	0.88 (0.37)	0.05	0.31 (0.16)	0.97 (0.15)	0.89 (0.37)
	50	0.30 (0.14)	0.96 (0.14)	0.89 (0.34)	0.04	0.30 (0.15)	0.96 (0.14)	0.90 (0.34)
	80	0.29 (0.12)	0.98 (0.12)	0.91 (0.30)	0.03	0.29 (0.13)	0.98 (0.12)	0.92 (0.31)
	100	0.29 (0.12)	0.99 (0.11)	0.92 (0.29)	0.03	0.29 (0.12)	0.99 (0.11)	0.92 (0.29)
	200	0.28 (0.09)	0.99 (0.08)	0.93 (0.23)	0.02	0.28 (0.09)	0.99 (0.08)	0.93 (0.24)

		MALS2				ACS			
σ^2_q	n	$\hat\beta_0$	$\hat\beta_1$	$\hat\beta_2$	$\hat\sigma^2_q$	$\hat\beta_0$	$\hat\beta_1$	$\hat\beta_2$	$\hat\sigma^2_q$
1	10	0.32 (0.59)	1.04 (0.60)	0.93 (1.27)	1.32	-0.54 (4533.46)	1.72 (8420.02)	2.23 (14774.21)	0.23
	20	0.18 (0.65)	1.09 (0.58)	1.22 (1.52)	0.95	0.10 (1772.95)	0.91 (1562.13)	0.56 (5679.04)	0.73
	30	0.13 (0.61)	1.05 (0.49)	1.36 (1.47)	0.92	-0.11 (520.64)	0.59 (243.21)	2.02 (1618.44)	0.82
	40	0.12 (0.58)	1.06 (0.43)	1.36 (1.36)	0.90	-0.04 (80.29)	1.14 (27.86)	1.74 (238.51)	0.80
	50	0.12 (0.53)	1.02 (0.39)	1.35 (1.28)	0.94	0.14 (321.12)	1.12 (138.48)	1.26 (1065.96)	0.82
	80	0.16 (0.40)	1.03 (0.29)	1.30 (0.97)	0.92	0.17 (8.52)	1.02 (2.68)	1.24 (22.50)	0.90
	100	0.18 (0.32)	1.01 (0.24)	1.19 (0.75)	0.95	0.12 (3.67)	1.02 (1.44)	1.36 (11.02)	0.90
	200	0.21 (0.20)	1.01 (0.16)	1.12 (0.46)	0.97	0.19 (0.21)	1.02 (0.16)	1.15 (0.49)	0.95
0	10	0.36 (0.18)	0.97 (0.21)	0.81 (0.37)	0.21	-0.69 (7683.91)	0.96 (3889.73)	3.39 (19212.55)	-0.44
	20	0.33 (0.18)	0.97 (0.19)	0.84 (0.39)	0.09	-0.64 (4551.52)	2.15 (5375.02)	4.02 (14637.91)	-0.42
	30	0.32 (0.16)	0.98 (0.17)	0.87 (0.37)	0.06	0.15 (838.11)	0.93 (355.76)	0.97 (2598.56)	-0.05
	40	0.31 (0.16)	0.97 (0.15)	0.88 (0.37)	0.05	-0.10 (199.51)	1.05 (55.75)	1.90 (510.64)	0.02
	50	0.30 (0.14)	0.96 (0.14)	0.89 (0.33)	0.04	0.11 (253.73)	1.07 (37.26)	1.36 (810.24)	-0.05
	80	0.29 (0.12)	0.98 (0.12)	0.91 (0.30)	0.03	0.12 (34.83)	1.09 (11.11)	1.36 (91.89)	-0.05
	100	0.29 (0.12)	0.98 (0.11)	0.92 (0.29)	0.03	0.13 (1.17)	1.03 (0.57)	1.32 (3.46)	-0.04
	200	0.28 (0.09)	0.99 (0.08)	0.93 (0.23)	0.02	0.20 (0.14)	1.00 (0.12)	1.15 (0.36)	0.20

hypothesis testing. In practice, we often encounter small to moderate sample sizes. In this situation, we strongly recommend to use the modified versions of estimators, namely, MALS1, MALS2 and MCS estimators and their associated tests, depending on the presence or absence of an equation error.

C Additional Example

Table 6: Estimates (standard errors) for the quadratic model with equation error ($\sigma_q^2 = 1$).

		OLS				ACS			
κ_a	n	$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\sigma}_q^2$	$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\sigma}_q^2$
0.97	10	0.23 (0.44)	1.00 (0.39)	0.99 (0.67)	0.65	0.22 (0.48)	1.01 (0.43)	1.12 (0.84)	0.79
	20	0.29 (0.34)	0.97 (0.33)	0.90 (0.56)	0.84	0.24 (0.36)	1.01 (0.34)	1.06 (0.64)	0.80
	30	0.28 (0.28)	0.97 (0.28)	0.92 (0.48)	0.89	0.24 (0.30)	1.01 (0.29)	1.03 (0.55)	0.86
	40	0.28 (0.25)	0.99 (0.25)	0.90 (0.43)	0.93	0.24 (0.26)	1.01 (0.26)	1.03 (0.49)	0.91
	50	0.28 (0.23)	0.98 (0.23)	0.90 (0.40)	0.96	0.25 (0.24)	1.00 (0.24)	1.01 (0.44)	0.92
	80	0.29 (0.18)	0.98 (0.19)	0.87 (0.31)	0.97	0.25 (0.19)	1.00 (0.19)	1.01 (0.36)	0.95
	100	0.28 (0.16)	0.98 (0.17)	0.90 (0.29)	0.99	0.25 (0.17)	1.00 (0.17)	1.01 (0.32)	0.96
	200	0.29 (0.11)	0.98 (0.12)	0.89 (0.20)	1.01	0.25 (0.12)	1.00 (0.12)	1.00 (0.23)	0.98
0.91	10	0.31 (0.44)	0.96 (0.40)	0.82 (0.65)	0.67	0.10 (18.89)	1.07 (13.09)	1.43 (56.09)	0.81
	20	0.32 (0.34)	0.96 (0.34)	0.77 (0.52)	0.89	0.20 (0.60)	1.06 (0.50)	1.31 (1.33)	0.85
	30	0.34 (0.28)	0.94 (0.29)	0.74 (0.44)	0.95	0.20 (0.37)	1.02 (0.33)	1.15 (0.74)	0.83
	40	0.34 (0.25)	0.93 (0.25)	0.72 (0.39)	0.98	0.21 (0.31)	1.02 (0.29)	1.12 (0.63)	0.87
	50	0.34 (0.22)	0.94 (0.23)	0.71 (0.35)	1.01	0.22 (0.28)	1.03 (0.26)	1.09 (0.56)	0.90
	80	0.35 (0.18)	0.93 (0.19)	0.70 (0.28)	1.02	0.23 (0.22)	1.01 (0.21)	1.06 (0.44)	0.94
	100	0.35 (0.16)	0.92 (0.17)	0.69 (0.25)	1.04	0.23 (0.19)	1.00 (0.19)	1.05 (0.39)	0.95
	200	0.36 (0.11)	0.92 (0.12)	0.68 (0.18)	1.05	0.25 (0.14)	1.00 (0.13)	1.02 (0.28)	0.98
0.87	10	0.34 (0.44)	0.90 (0.42)	0.74 (0.64)	0.70	-0.33 (246.05)	0.98 (128.88)	2.85 (686.36)	12.12
	20	0.35 (0.33)	0.90 (0.34)	0.69 (0.50)	0.89	0.08 (1.91)	1.02 (1.32)	1.43 (5.10)	1.04
	30	0.36 (0.28)	0.92 (0.29)	0.67 (0.41)	0.96	0.12 (0.96)	1.02 (0.76)	1.35 (2.57)	1.48
	40	0.37 (0.24)	0.90 (0.25)	0.64 (0.37)	0.98	0.17 (0.37)	1.04 (0.32)	1.22 (0.79)	0.85
	50	0.38 (0.22)	0.90 (0.23)	0.61 (0.34)	1.02	0.19 (0.32)	1.02 (0.29)	1.17 (0.67)	0.88
	80	0.39 (0.18)	0.89 (0.18)	0.60 (0.27)	1.05	0.22 (0.24)	1.03 (0.22)	1.08 (0.51)	0.93
	100	0.38 (0.16)	0.90 (0.17)	0.60 (0.24)	1.06	0.22 (0.22)	1.01 (0.20)	1.08 (0.45)	0.95
	200	0.38 (0.11)	0.89 (0.12)	0.60 (0.17)	1.07	0.24 (0.15)	1.01 (0.14)	1.04 (0.32)	0.97
0.83	10	0.32 (0.46)	0.95 (0.43)	0.73 (0.65)	0.76	0.07 (69.23)	0.87 (27.72)	1.50 (204.52)	20.48
	20	0.39 (0.33)	0.89 (0.33)	0.60 (0.47)	0.89	0.34 (62.83)	0.90 (55.71)	0.66 (179.84)	9.07
	30	0.39 (0.28)	0.88 (0.29)	0.58 (0.40)	1.00	0.15 (9.12)	1.06 (6.51)	1.64 (17.93)	0.98
	40	0.41 (0.24)	0.87 (0.25)	0.55 (0.35)	1.03	0.17 (1.68)	1.05 (0.74)	1.23 (4.49)	0.88
	50	0.40 (0.22)	0.87 (0.23)	0.55 (0.31)	1.06	0.20 (1.60)	1.00 (1.21)	1.13 (4.53)	0.92
	80	0.41 (0.17)	0.87 (0.18)	0.52 (0.25)	1.08	0.20 (0.45)	1.03 (0.28)	1.13 (1.08)	0.92
	100	0.41 (0.16)	0.86 (0.16)	0.52 (0.22)	1.07	0.21 (0.25)	1.01 (0.22)	1.13 (0.56)	0.93
	200	0.42 (0.11)	0.86 (0.12)	0.50 (0.16)	1.11	0.23 (0.17)	1.01 (0.15)	1.06 (0.38)	0.97
0.77	10	0.38 (0.45)	0.89 (0.43)	0.60 (0.61)	0.75	-0.51 (1428.30)	1.98 (808.93)	2.14 (3185.92)	298.75
	20	0.41 (0.34)	0.83 (0.33)	0.53 (0.44)	0.95	-0.92 (143.20)	1.01 (77.32)	5.11 (419.06)	1.10
	30	0.44 (0.28)	0.83 (0.28)	0.45 (0.36)	1.00	0.30 (125.44)	0.70 (134.93)	0.92 (387.37)	1.72
	40	0.42 (0.24)	0.82 (0.25)	0.46 (0.31)	1.06	-0.22 (36.42)	0.95 (20.90)	2.24 (101.05)	1.20
	50	0.44 (0.22)	0.83 (0.22)	0.45 (0.28)	1.09	0.07 (20.11)	0.97 (8.63)	1.61 (66.67)	0.86
	80	0.44 (0.17)	0.81 (0.18)	0.43 (0.23)	1.11	0.18 (11.41)	1.11 (2.92)	1.26 (26.75)	0.88
	100	0.44 (0.16)	0.81 (0.16)	0.43 (0.20)	1.12	0.20 (0.54)	1.03 (0.33)	1.12 (0.91)	0.94
	200	0.45 (0.11)	0.81 (0.12)	0.41 (0.15)	1.14	0.20 (0.22)	1.02 (0.17)	1.15 (0.51)	0.95

Example: Data used in Dear et al. (1997). Dear et al. (1997) investigated datasets from a WHO MONICA sub-population that has the typical pattern observed in the complete population. The dataset consists of means and standard deviations of estimated changes over a five year period in rates of coronary heart disease and four different risk factors for the disease. These datasets are also analyzed in Kulathinal et al. (2002). Both papers were interested in the point and interval estimation of the correlation, where the linear model was used. Here we are interested in the risk factor of total cholesterol (covariate) versus coronary disease (response).

The scatter plot of change in coronary death rate and change in cholesterol (Dear et al., 1997, p. 2179) shows that there is a possibility of a nonlinear relation. The (estimated) average reliability ratio is 0.83, which is not too bad

Table 7: Rejection rate of the null hypothesis $H_0 : \beta_2 = 0$ in quadratic model. ($\kappa_a = 0.77$)

		$\sigma_q^2 = 1$						$\sigma_q^2 = 0$					
		Wald-type			Score-type			Wald-type			Score-type		
n	β_2	MALS1	MALS2	MCS	MALS1	MALS2	MCS	MALS1	MALS2	MCS	MALS1	MALS2	MCS
10	-0.5	0.18	0.20	0.18	0.05	0.05	0.05	0.27	0.31	0.27	0.10	0.09	0.10
	-0.25	0.18	0.19	0.18	0.05	0.05	0.05	0.19	0.22	0.19	0.06	0.06	0.06
	0	0.15	0.16	0.15	0.05	0.05	0.05	0.17	0.20	0.17	0.05	0.05	0.05
	0.25	0.17	0.18	0.16	0.04	0.04	0.05	0.18	0.22	0.18	0.05	0.05	0.05
	0.5	0.18	0.20	0.18	0.06	0.06	0.05	0.26	0.30	0.26	0.09	0.08	0.09
20	-0.5	0.09	0.10	0.09	0.08	0.08	0.07	0.18	0.26	0.18	0.15	0.16	0.15
	-0.25	0.08	0.09	0.08	0.06	0.06	0.06	0.10	0.16	0.10	0.08	0.08	0.08
	0	0.08	0.09	0.08	0.06	0.06	0.06	0.08	0.12	0.08	0.05	0.05	0.06
	0.25	0.09	0.09	0.09	0.06	0.06	0.06	0.11	0.16	0.11	0.08	0.08	0.09
	0.5	0.09	0.10	0.09	0.07	0.07	0.07	0.18	0.25	0.18	0.15	0.16	0.15
30	-0.5	0.08	0.09	0.08	0.09	0.09	0.08	0.17	0.26	0.18	0.21	0.22	0.21
	-0.25	0.05	0.06	0.06	0.06	0.06	0.06	0.09	0.15	0.09	0.10	0.10	0.10
	0	0.04	0.05	0.04	0.05	0.05	0.05	0.05	0.09	0.05	0.06	0.05	0.05
	0.25	0.05	0.06	0.05	0.05	0.05	0.05	0.09	0.14	0.09	0.09	0.09	0.09
	0.5	0.09	0.09	0.08	0.08	0.08	0.07	0.18	0.27	0.18	0.20	0.21	0.20
50	-0.5	0.08	0.08	0.08	0.10	0.10	0.09	0.21	0.31	0.21	0.30	0.31	0.30
	-0.25	0.06	0.06	0.06	0.06	0.06	0.07	0.09	0.15	0.09	0.12	0.12	0.12
	0	0.05	0.06	0.05	0.06	0.06	0.06	0.04	0.08	0.04	0.06	0.06	0.05
	0.25	0.06	0.06	0.07	0.07	0.07	0.07	0.09	0.14	0.08	0.11	0.12	0.12
	0.5	0.08	0.08	0.08	0.10	0.10	0.10	0.19	0.30	0.20	0.27	0.29	0.27
100	-0.5	0.13	0.13	0.12	0.16	0.16	0.15	0.35	0.44	0.35	0.45	0.47	0.46
	-0.25	0.06	0.06	0.07	0.07	0.07	0.07	0.13	0.18	0.13	0.16	0.17	0.16
	0	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.07	0.05	0.05	0.06	0.06
	0.25	0.07	0.07	0.07	0.09	0.09	0.08	0.14	0.18	0.13	0.16	0.17	0.16
	0.5	0.15	0.15	0.14	0.17	0.17	0.17	0.36	0.44	0.35	0.47	0.48	0.47
200	-0.5	0.24	0.24	0.23	0.28	0.28	0.28	0.65	0.68	0.65	0.73	0.74	0.73
	-0.25	0.09	0.09	0.10	0.10	0.10	0.11	0.24	0.26	0.24	0.27	0.27	0.27
	0	0.05	0.05	0.05	0.05	0.05	0.05	0.06	0.06	0.06	0.05	0.05	0.05
	0.25	0.10	0.10	0.10	0.12	0.12	0.11	0.24	0.26	0.24	0.27	0.27	0.26
	0.5	0.24	0.24	0.23	0.27	0.27	0.27	0.63	0.65	0.62	0.71	0.72	0.71

Table 8: Rejection rate of the null hypothesis $H_0 : \beta_3 = 0$ in cubic model. ($\kappa_a = 0.93$)

		$\sigma_q^2 = 3$						$\sigma_q^2 = 0$					
		Wald-type			Score-type			Wald-type			Score-type		
n	β_3	MALS1	MALS2	MCS	MALS1	MALS2	MCS	MALS1	MALS2	MCS	MALS1	MALS2	MCS
10	-1	0.63	0.64	0.63	0.43	0.43	0.42	0.86	0.87	0.86	0.74	0.74	0.73
	-0.5	0.36	0.36	0.36	0.20	0.20	0.20	0.72	0.73	0.72	0.56	0.57	0.54
	0	0.20	0.20	0.21	0.09	0.09	0.09	0.22	0.23	0.22	0.09	0.10	0.10
	0.5	0.37	0.37	0.38	0.19	0.19	0.20	0.63	0.64	0.63	0.41	0.41	0.40
	1	0.58	0.58	0.57	0.37	0.37	0.36	0.75	0.75	0.75	0.60	0.60	0.58
20	-1	0.61	0.62	0.59	0.63	0.63	0.62	0.86	0.87	0.85	0.85	0.85	0.84
	-0.5	0.28	0.28	0.27	0.27	0.27	0.25	0.75	0.77	0.73	0.68	0.69	0.67
	0	0.09	0.09	0.08	0.07	0.07	0.07	0.16	0.17	0.16	0.13	0.13	0.12
	0.5	0.28	0.29	0.27	0.28	0.28	0.27	0.61	0.62	0.60	0.60	0.59	0.58
	1	0.58	0.58	0.55	0.59	0.59	0.57	0.77	0.77	0.76	0.76	0.76	0.76
30	-1	0.64	0.64	0.62	0.76	0.76	0.75	0.89	0.89	0.87	0.93	0.93	0.92
	-0.5	0.28	0.28	0.27	0.34	0.34	0.33	0.75	0.77	0.74	0.75	0.76	0.74
	0	0.07	0.07	0.07	0.07	0.07	0.08	0.12	0.14	0.13	0.11	0.11	0.11
	0.5	0.26	0.26	0.25	0.35	0.35	0.35	0.63	0.64	0.62	0.69	0.69	0.67
	1	0.58	0.59	0.55	0.71	0.71	0.69	0.80	0.80	0.78	0.86	0.86	0.85
50	-1	0.75	0.76	0.72	0.91	0.91	0.88	0.92	0.93	0.92	0.97	0.97	0.97
	-0.5	0.37	0.37	0.34	0.48	0.48	0.46	0.79	0.81	0.79	0.83	0.84	0.82
	0	0.06	0.06	0.05	0.06	0.06	0.06	0.12	0.14	0.12	0.11	0.10	0.11
	0.5	0.35	0.35	0.32	0.53	0.53	0.51	0.69	0.70	0.66	0.78	0.78	0.77
	1	0.67	0.67	0.64	0.85	0.85	0.83	0.84	0.84	0.83	0.94	0.94	0.92
100	-1	0.89	0.89	0.88	0.98	0.98	0.97	0.94	0.94	0.94	0.99	0.99	0.99
	-0.5	0.61	0.61	0.59	0.71	0.71	0.69	0.86	0.87	0.85	0.93	0.93	0.93
	0	0.07	0.07	0.07	0.06	0.06	0.06	0.11	0.13	0.11	0.08	0.08	0.08
	0.5	0.51	0.51	0.49	0.70	0.70	0.69	0.76	0.77	0.77	0.87	0.87	0.87
	1	0.85	0.85	0.83	0.95	0.95	0.95	0.90	0.91	0.90	0.97	0.97	0.98
200	-1	0.97	0.97	0.97	1	1	1	0.97	0.97	0.97	1	1	1
	-0.5	0.83	0.83	0.82	0.89	0.89	0.88	0.93	0.93	0.93	0.98	0.98	0.97
	0	0.07	0.07	0.08	0.05	0.05	0.05	0.11	0.11	0.10	0.08	0.07	0.08
	0.5	0.76	0.76	0.73	0.86	0.86	0.83	0.86	0.86	0.85	0.94	0.94	0.94
	1	0.93	0.93	0.93	0.98	0.98	0.98	0.95	0.95	0.94	0.99	0.99	0.99

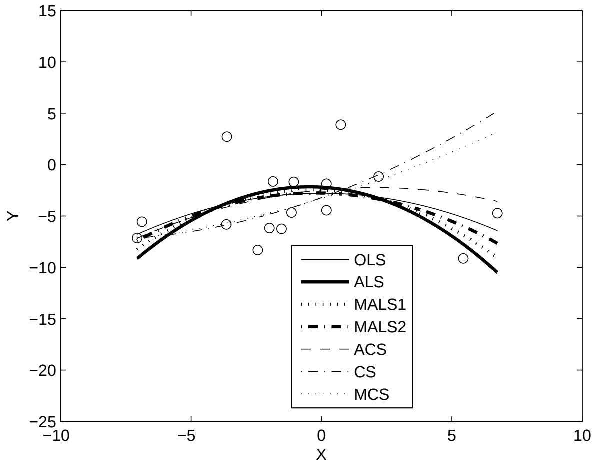

Figure 1: Quadratic regressions for data in Dear *et al.* (1997) data set.

for fitting a quadratic model. We fit the dataset with a quadratic model first and test if $H_0 : \beta_2 = 0$ is suitable or not. Table 9 displays the estimates of regression parameters and equation error variance and their associated test statistics. For the estimation part, it is clear that OLS is quite different from the others. Because the average error variances in the response variable ($\overline{\sigma_\varepsilon^2}$) is quite large, both ALS and MALS1 produce negative estimates of the equation error variance, only MALS2 and ACS produce positive estimates. Basically, the estimates show three different regression curves, namely, ALS, CS and ACS because ALS and MALSs are close and so are CS and MCS. The quadratic curves are shown in Figure 1.

The test statistics show that the null hypothesis is not rejected except for the Wald-type statistics induced by OLS and MALS2. However, the score test statistics for ALS and MALSs are very close to the critical value. Should we change the level just a bit from 0.05 to 0.075 or so, the null hypothesis would be rejected.

Now we use a linear model to fit the data. Table 10 shows the estimates. It turns out that only MALS2, ACS, and ML (via EM algorithm) result in a positive estimate of the equation error variance. Also OLS is close to MALS but this is merely a coincidence. Because ALS and MALSs are very close and

Table 9: Dear *et al.* (1997) data set: Estimates (standard errors) and test statistics for the quadratic model. ($H_0 : \beta_2 = 0$, $\chi^2_{1,0.95} = 3.84$)

Estimates	OLS	ALS	MALS1	MALS2	ACS	CS	MCS
$\hat{\beta}_0$	-2.79 (1.06)	-2.20 (1.45)	-2.47 (1.22)	-2.77 (1.07)	-2.47 (1.03)	-3.22 (0.99)	-3.32 (0.91)
$\hat{\beta}_1$	0.00 (0.19)	-0.15 (0.35)	-0.10 (0.30)	-0.05 (0.24)	0.24 (0.33)	0.91 (1.02)	0.76 (0.75)
$\hat{\beta}_2$	-0.08 (0.03)	-0.16 (0.10)	-0.13 (0.07)	-0.10 (0.05)	-0.06 (0.05)	0.05 (0.13)	0.03 (0.10)
$\hat{\sigma}_q^2$	-5.39	-6.99	-4.15	5.41	5.44		

Test statistics	Wald-type	Score-type
OLS	6.28	3.80
ALS	2.57	3.57
MALS1	3.47	3.60
MALS2	4.87	3.64
ACS	1.64	-
CS	0.14	0.23
MCS	0.12	0.17

$n = 16$, $\overline{\sigma_\delta^2} = 2.40$ ($\max \sigma_{\delta i}^2 = 6.20$, $\min \sigma_{\delta i}^2 = 1.06$), $\overline{\sigma_\varepsilon^2} = 16.12$ ($\max \sigma_{\varepsilon i}^2 = 126.11$, $\min \sigma_{\varepsilon i}^2 = 0.59$), $\hat{\sigma}^2 = 11.41$ and $\hat{\kappa}_a = 0.83$.

Table 10: Dear *et al.* (1997) data set: Estimates (standard errors) for the linear model.

Estimates	OLS	ALS	MALS1	MALS2
$\hat{\beta}_0$	-3.83 (0.95)	-3.82 (0.97)	-3.82 (0.97)	-3.83 (0.96)
$\hat{\beta}_1$	0.04 (0.24)	0.05 (0.29)	0.05 (0.28)	0.05 (0.27)
$\hat{\sigma}_q^2$	-3.51	-3.52	-1.72	3.46

Estimates	ACS	CS	MCS	ML
$\hat{\beta}_0$	-2.72 (1.00)	-3.28 (0.90)	-3.33 (0.89)	-2.66 (0.90)
$\hat{\beta}_1$	0.47 (0.30)	0.66 (0.41)	0.62 (0.37)	0.48 (0.30)
$\hat{\sigma}_q^2$	4.98			5.06

CS and MCS are very close too, we only show the ALS, CS, ACS and ML regression lines in Figure 2.

Because this dataset is relatively small, $n = 16$ only, the analysis might not be very accurate. We can make better analyses for both point estimation and hypothesis testing if we can collect more data. However, the example can still serve our purpose to illustrate that our proposed ALS and its modification (MALS) can be used to deal with a polynomial model with small sample size.

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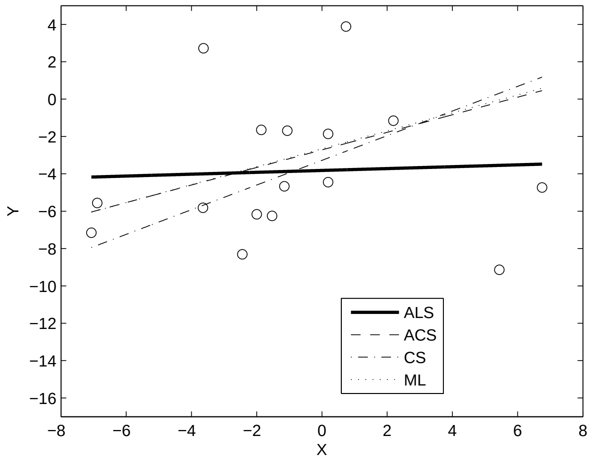

Figure 2: Linear regressions for data in Dear *et al.* (1997) data set.

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